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Frequency domain analysis of linear circuits using synchronous detection

Physics 401, Spring 2013
Eugene V. Colla

Outline

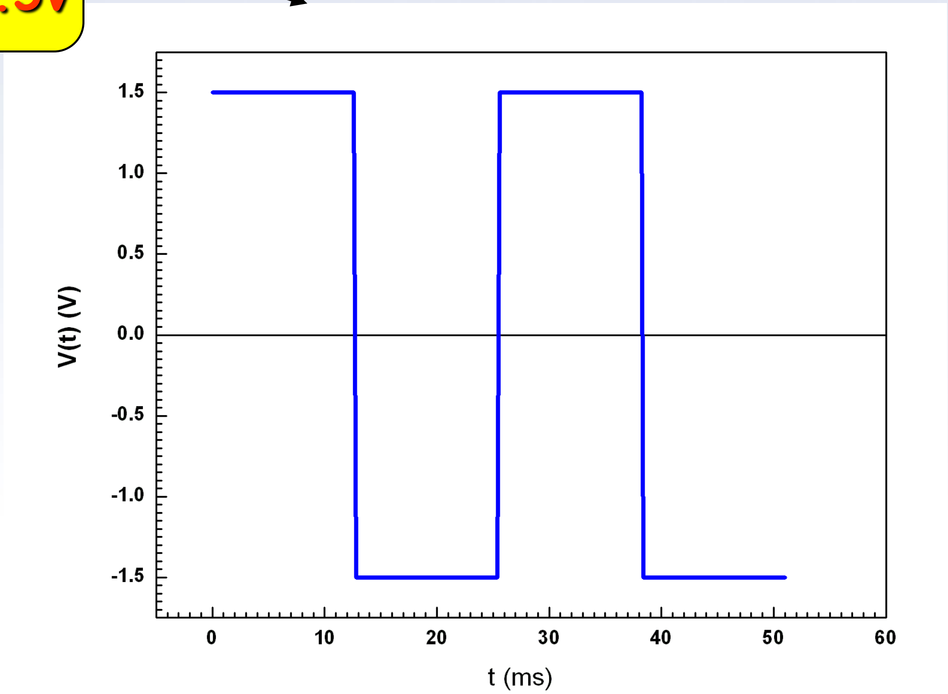
1. Fourier transform. Discrete Fourier transform. Some properties.
2. Time domain and Frequency domain representation of the data.
3. Frequency domain spectroscopy (FDS)
4. Lock-in amplifiers
5. Practical application of lock-in's in FDS
6. Taking data and simple data analysis using OriginPro.

in 1822, Jean Baptiste Fourier developed the theory that shows that any real waveform can be represented by the sum of sinusoidal waves.

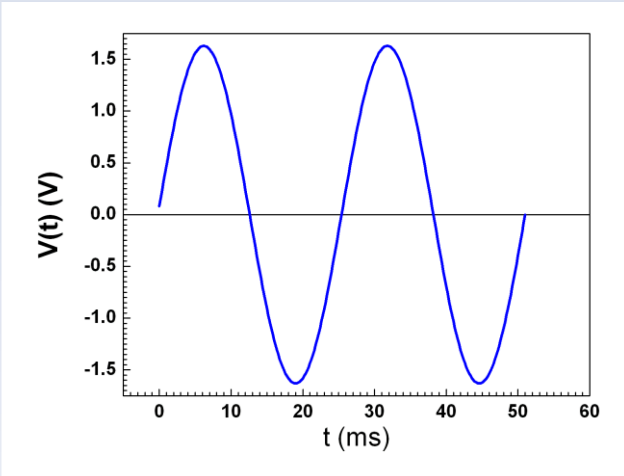
Jean Baptiste Joseph Fourier
(1768 – 1830)

Let we try to create the square wave as a sum of sine waves of different frequencies

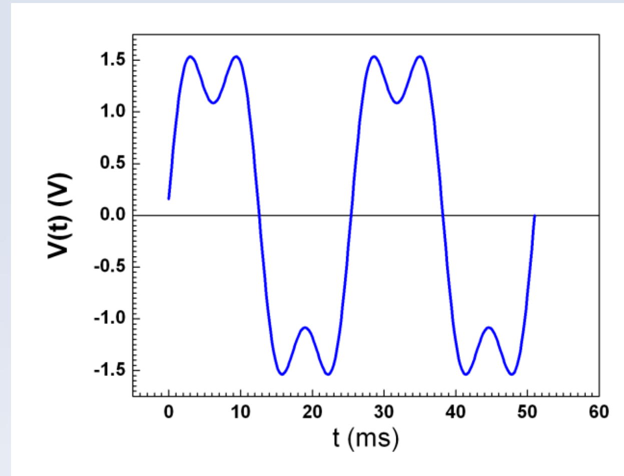
Square wave.
F=40Hz, A=1.5V



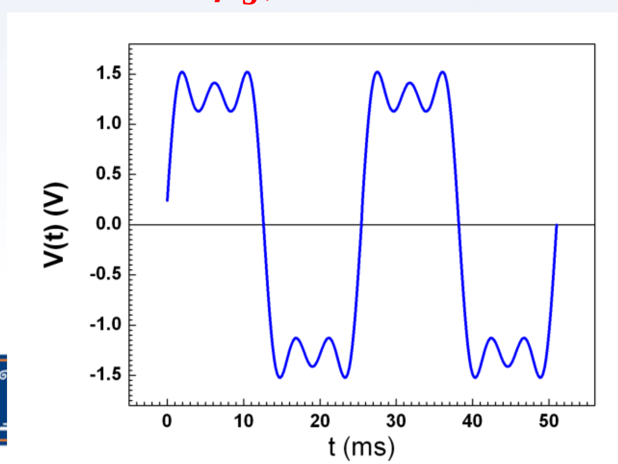
$$A_1\sin(2\pi\omega t)$$



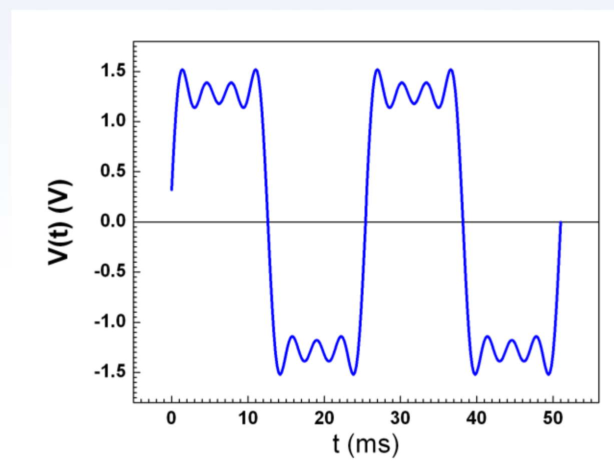
$$A_1\sin(2\pi\omega t)+A_3\sin(2\pi 3\omega t+\varphi_3)$$



$$A_1\sin(2\pi\omega t)+A_3\sin(2\pi 3\omega t+\varphi_3)+$$
$$A_5\sin(2\pi 5\omega t+\varphi_5)$$



$$A_1\sin(2\pi\omega t)+A_3\sin(2\pi 3\omega t+\varphi_3)+$$
$$A_5\sin(2\pi 5\omega t+\varphi_5)+A_7\sin(2\pi 7\omega t+\varphi_7)$$



Discrete Fourier Transform

The continues Fourier transformation of the signal h(t) can be written as:

$$H(f)=\int_{-\infty}^{+\infty} h(t)e^{2\pi jf}dt; \quad j=\sqrt{-1}$$

H(f) represents in frequency domain mode the time domain signal h(t)

Equation for inverse Fourier transform gives the correspondence of the infinite continues frequency spectra to the corresponding time domain signal.

$$h(t)=\int_{-\infty}^{+\infty} H(f)e^{-2\pi jf}df$$

In real life we working with discrete representation of the time domain signal recorded during a finite time.

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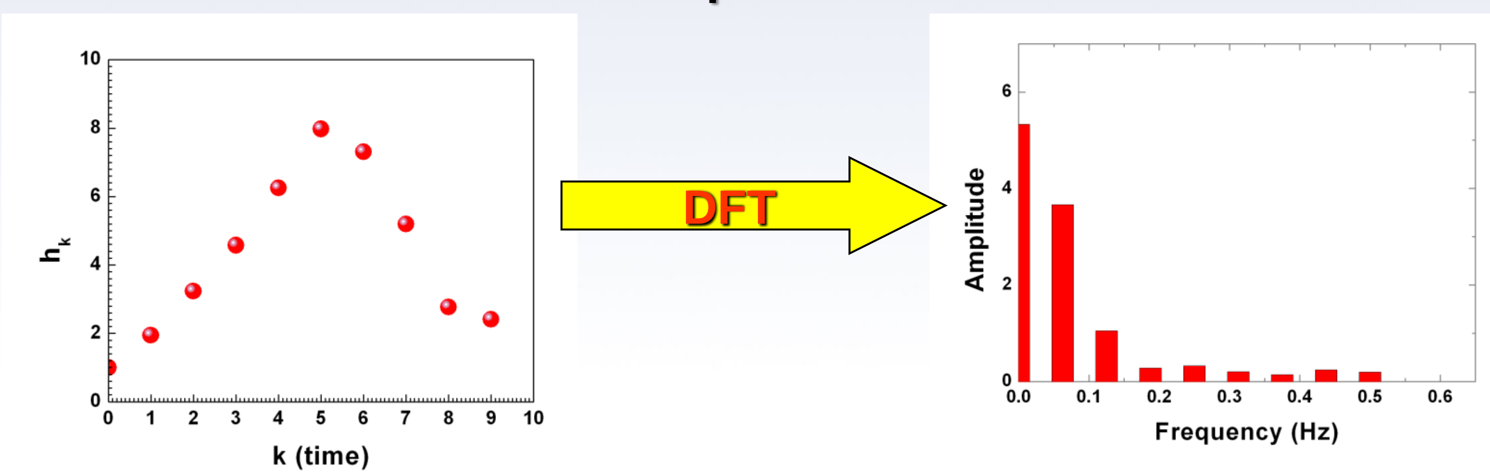
Discrete Fourier Transform

It comes out that in practice more useful is the representation the frequency domain pattern of the time domain signal h_k as sum of the frequency harmonic calculated as:

$$H_n(k) \cong \frac{1}{N}\sum_{k=0}^{N-1} h_k e^{j2\pi kn/N} \quad \text{and} \quad h_k \cong \frac{1}{N}\sum_{k=0}^{N-1} H_n(k) e^{j2\pi kn/N}$$

N – number of collected points



Time domain

Frequency domain

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Discrete Fourier Transform

For periodic signals with period T_0:

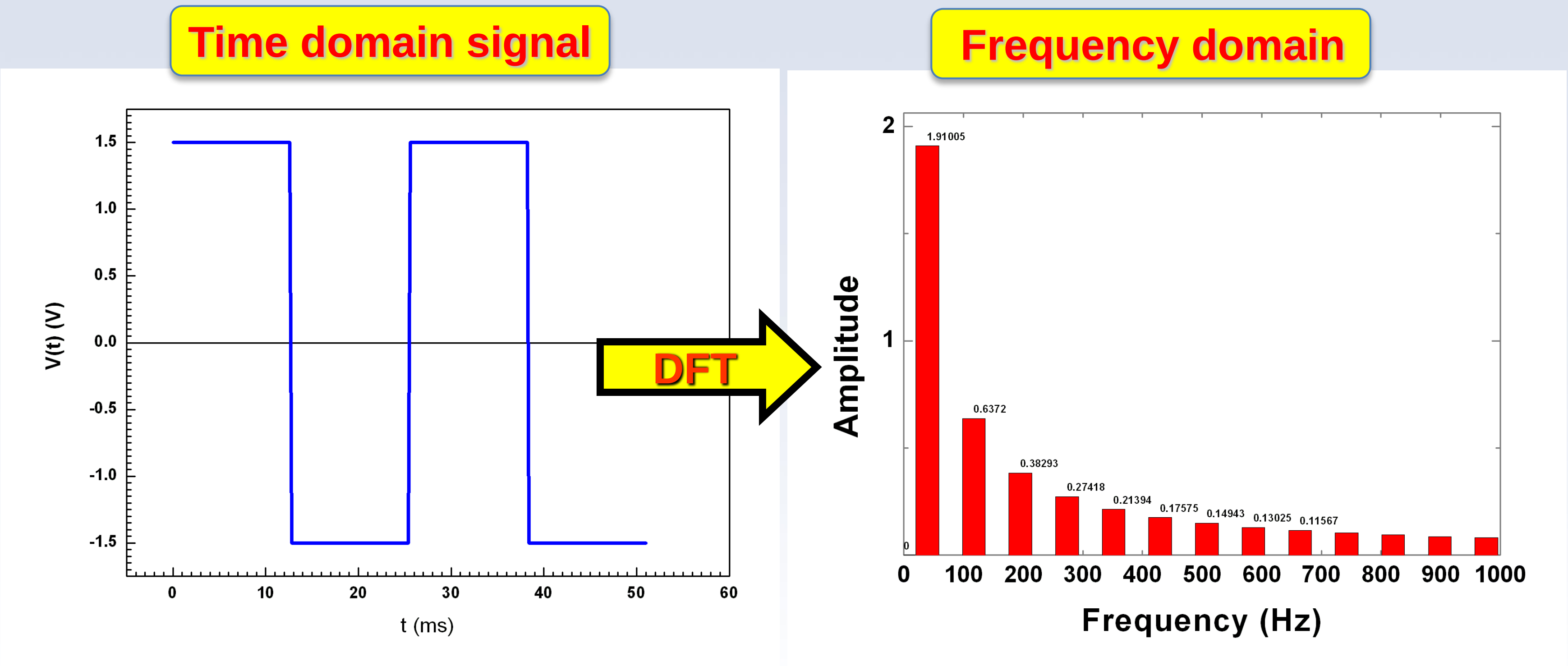
$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{T_0}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nt}{T_0}\right)$$

$$a_n = \frac{2}{T_0}\int_0^{T_0} F(t)\cos\left(\frac{2\pi nt}{T_0}\right)dt; \qquad b_n = \frac{2}{T_0}\int_0^{T_0} F(t)\cos\left(\frac{2\pi nt}{T_0}\right)dt;$$

$$a_0 = \frac{2}{T_0}\int_0^{T_0} F(t)dt;$$

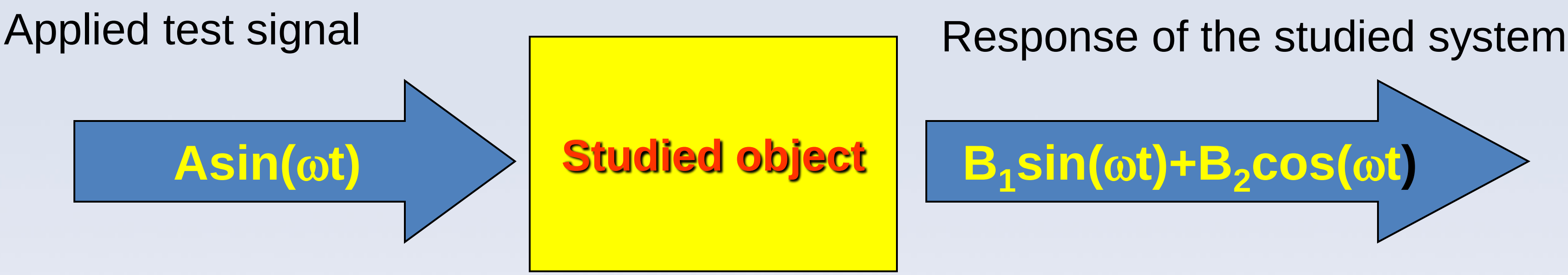
Discrete Fourier Transform

Now how we can find the amplitudes of the harmonics to compose the square wave signal from sine waves of different frequencies.

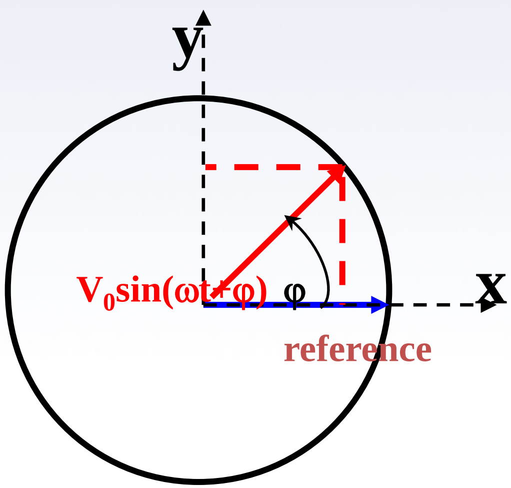


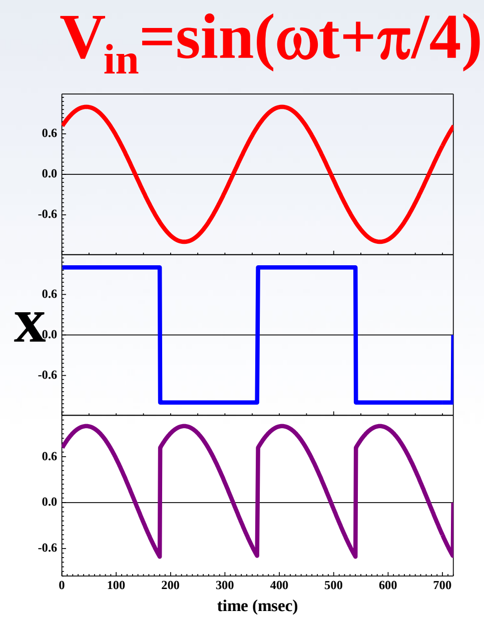
Decomposition the signal into the sine wave harmonics. The only modulus's of the harmonics amplitudes are presented in this picture.

Frequency Domain Spectroscopy
(linear system)



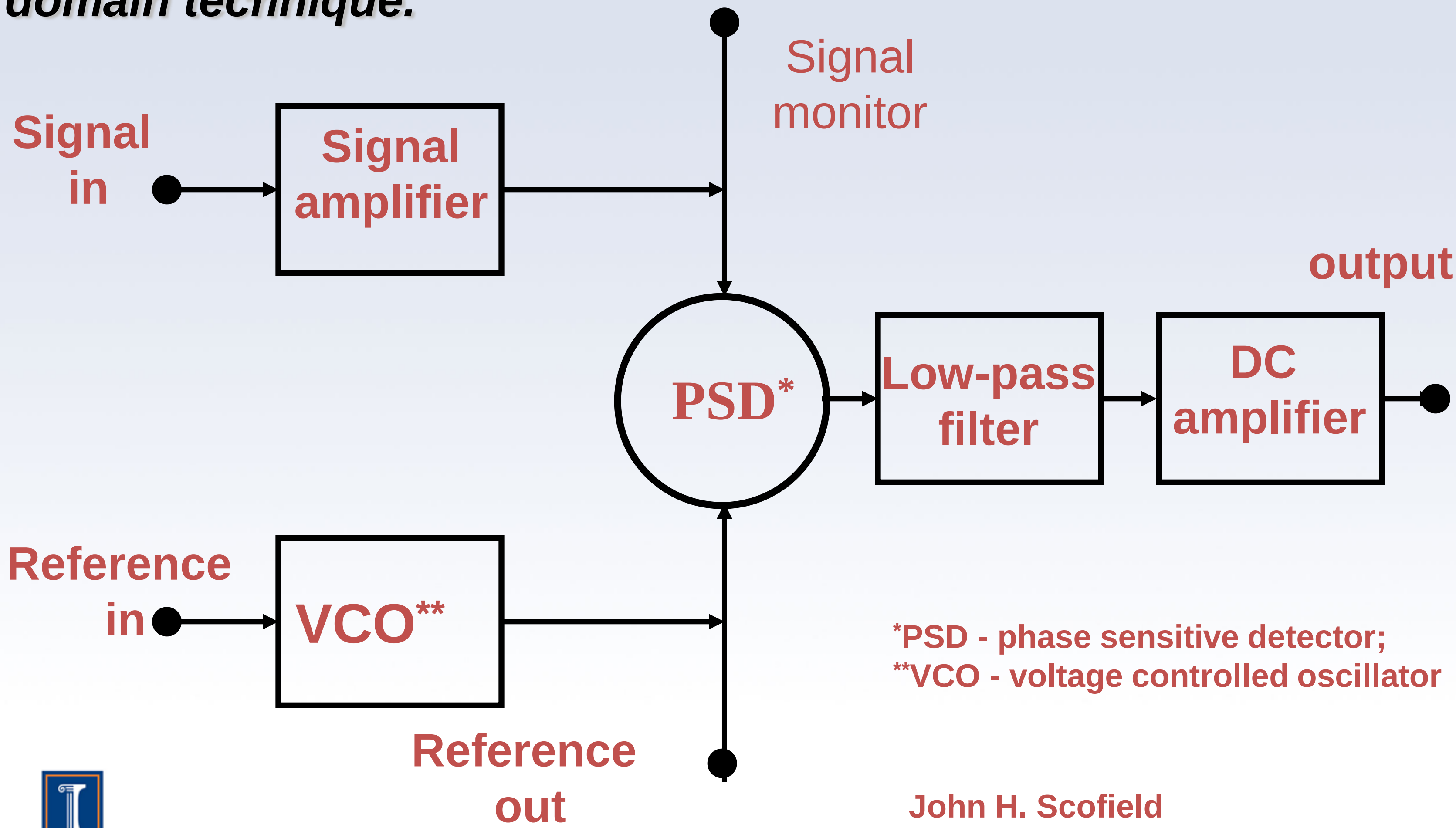
We applying the sine wave signal to tested object and measuring the response. Varying the frequency we can study the frequency properties of the system





Lock-in amplifier

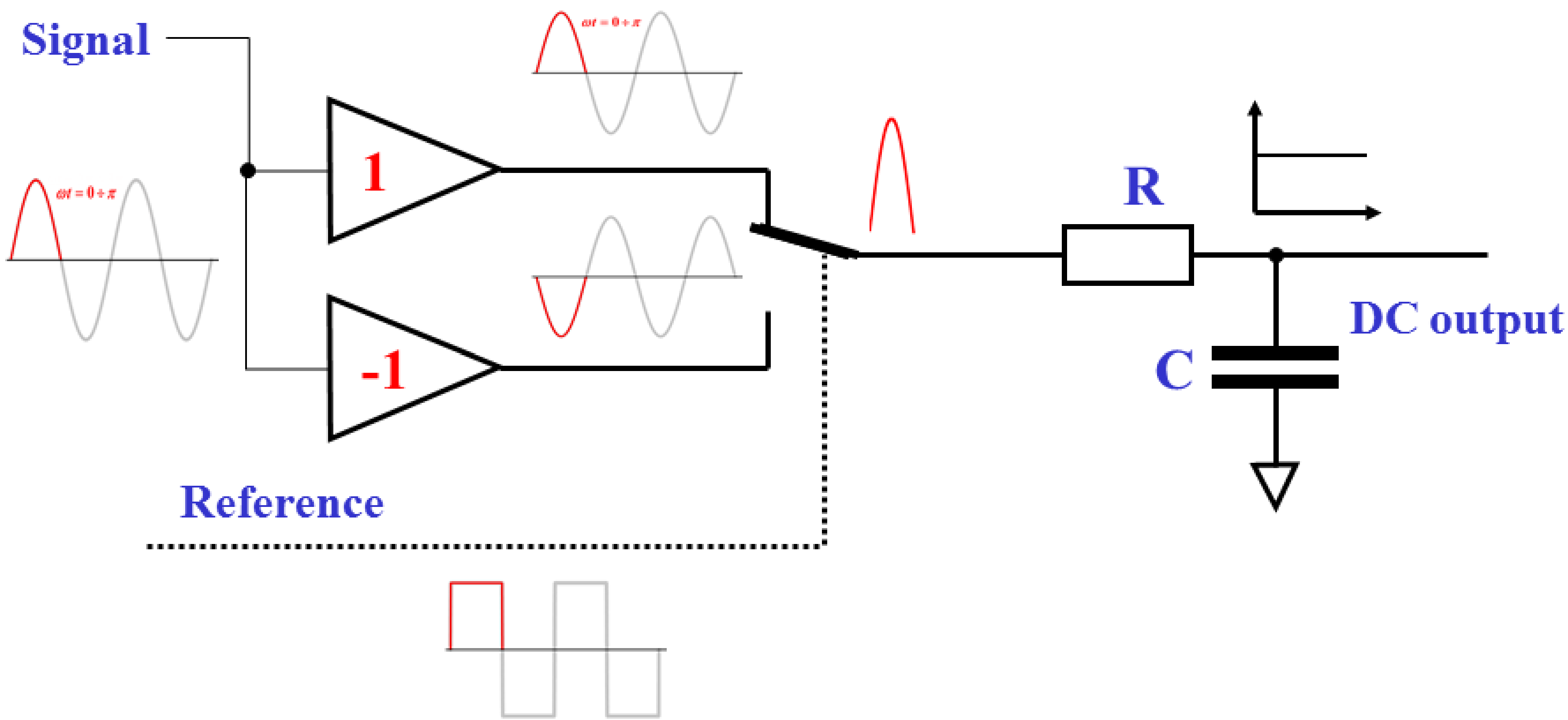
Now about the most powerful tool which can be used in frequency domain technique.



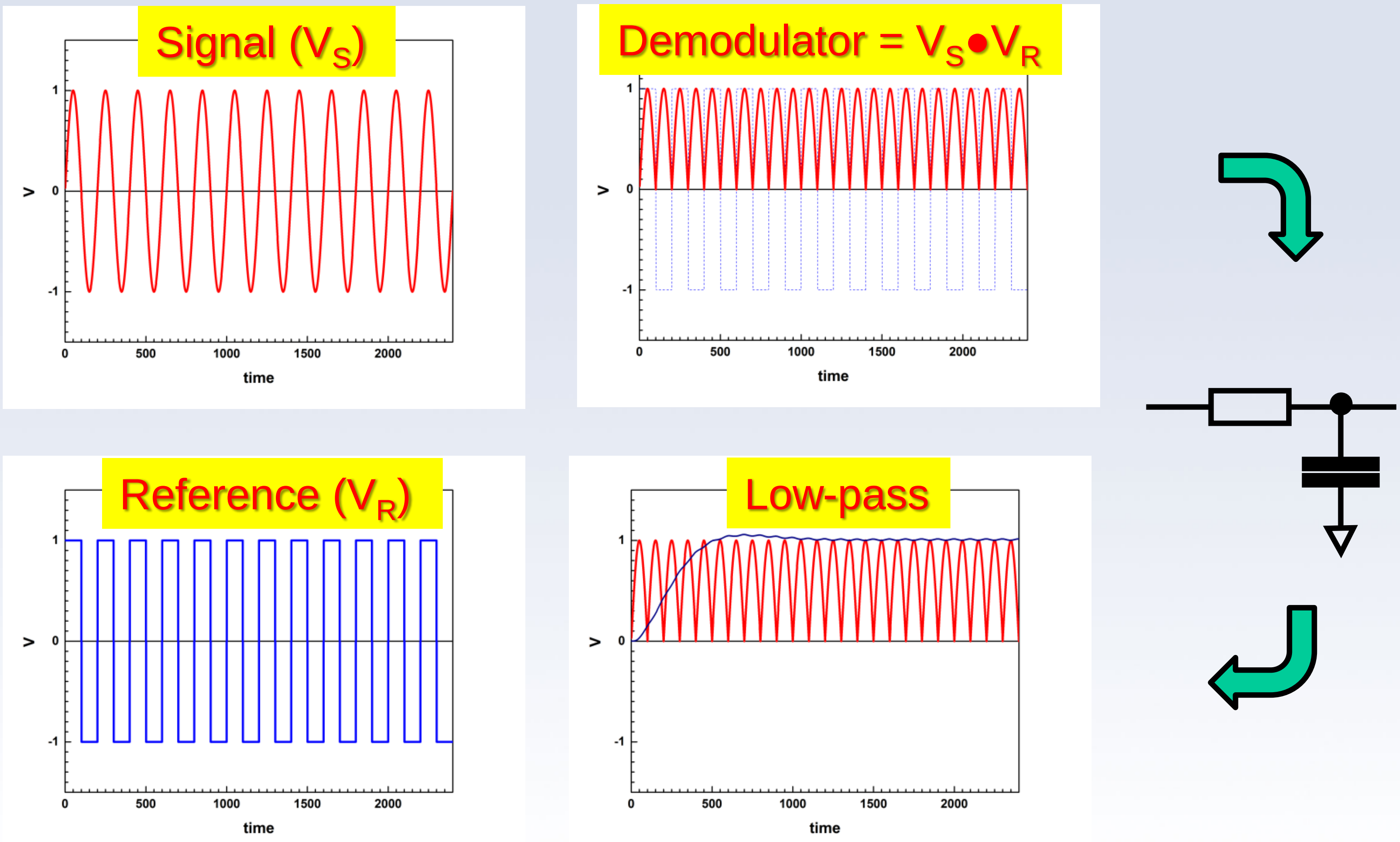
*PSD - phase sensitive detector;
**VCO - voltage controlled oscillator

John H. Scofield
American Journal of Physics 62 (2) 129-133 (Feb. 1994).

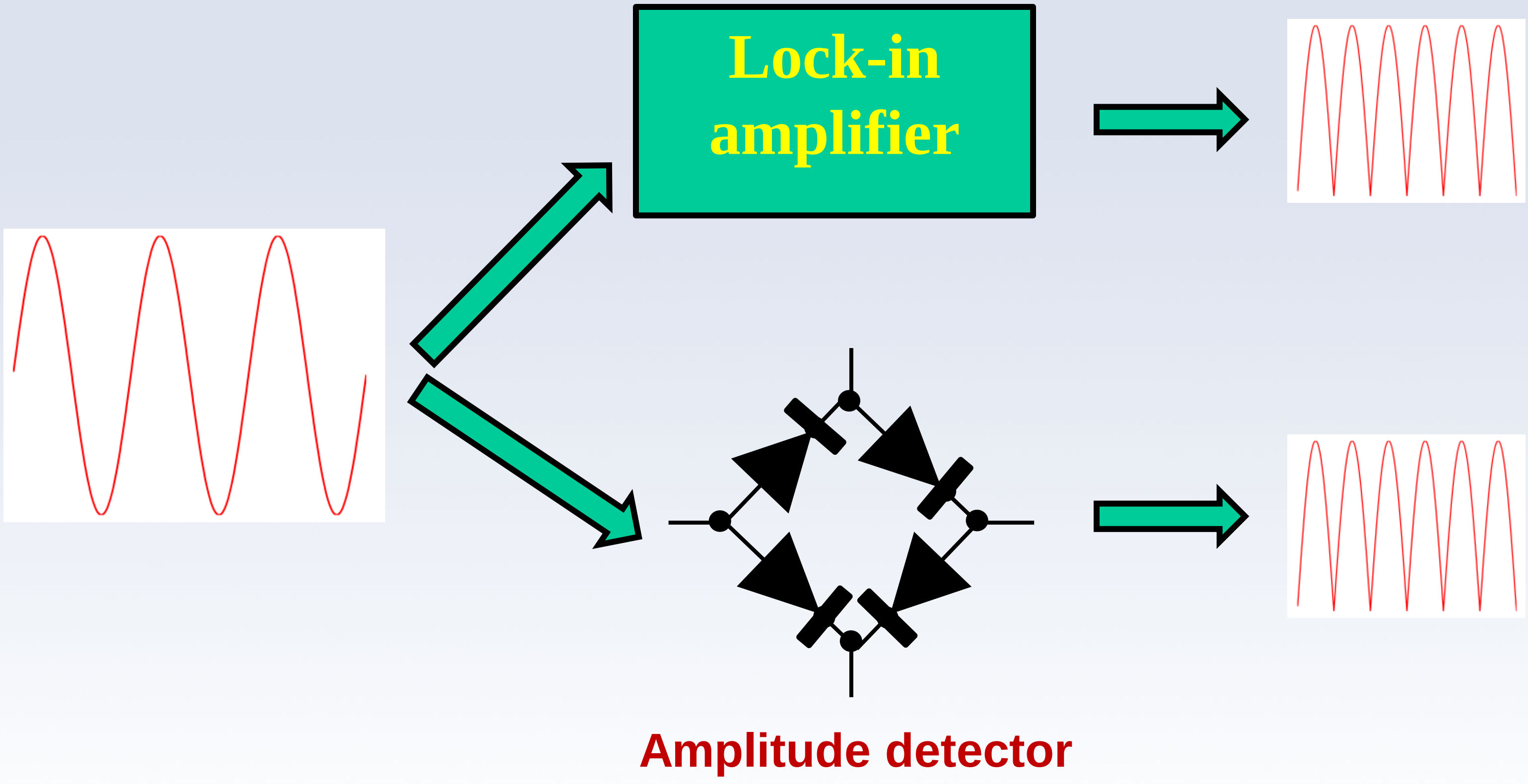
Lock-in amplifier. How it works.



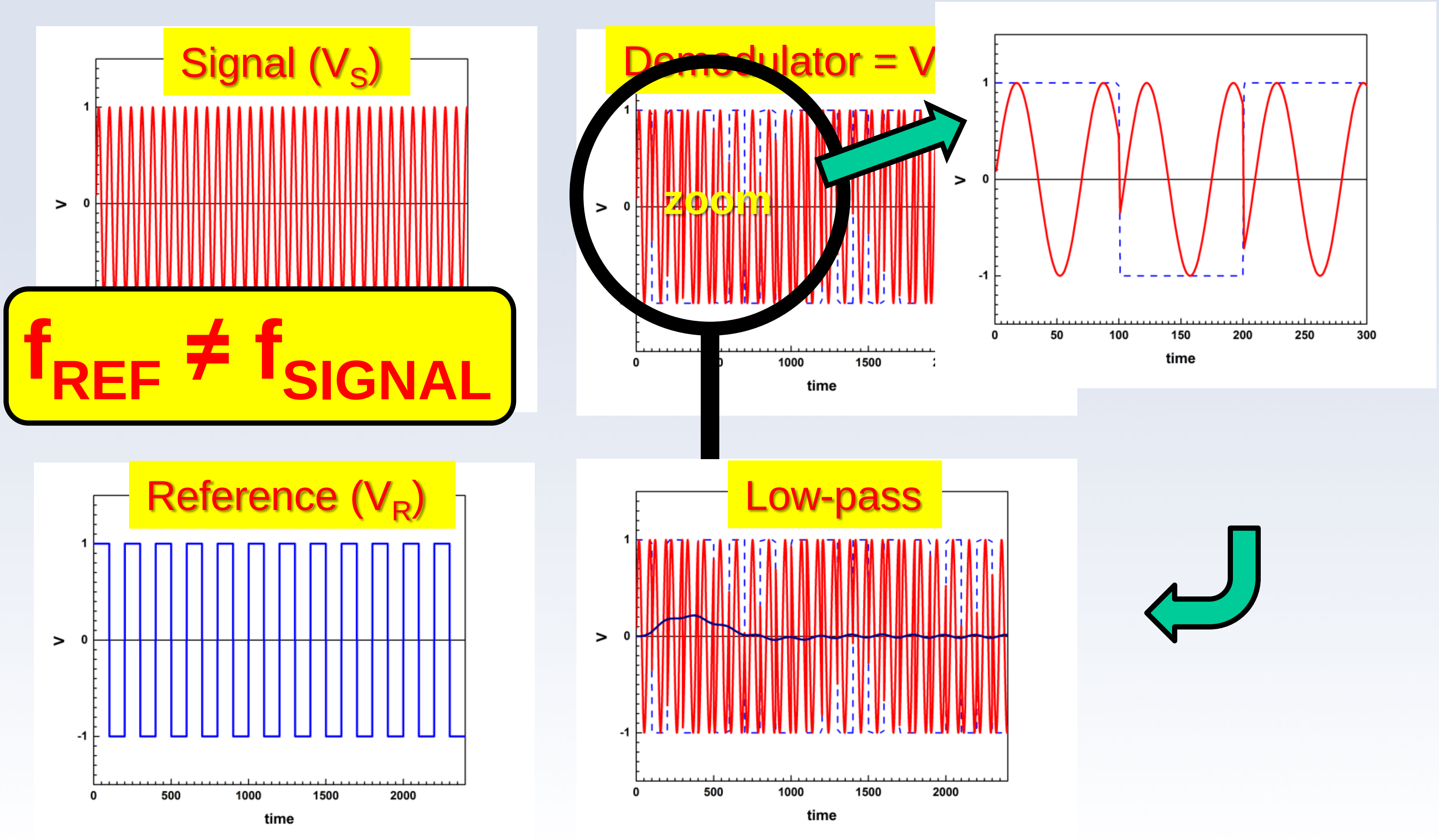
Lock-in amplifier. How it works.



Why lock-in amplifier?

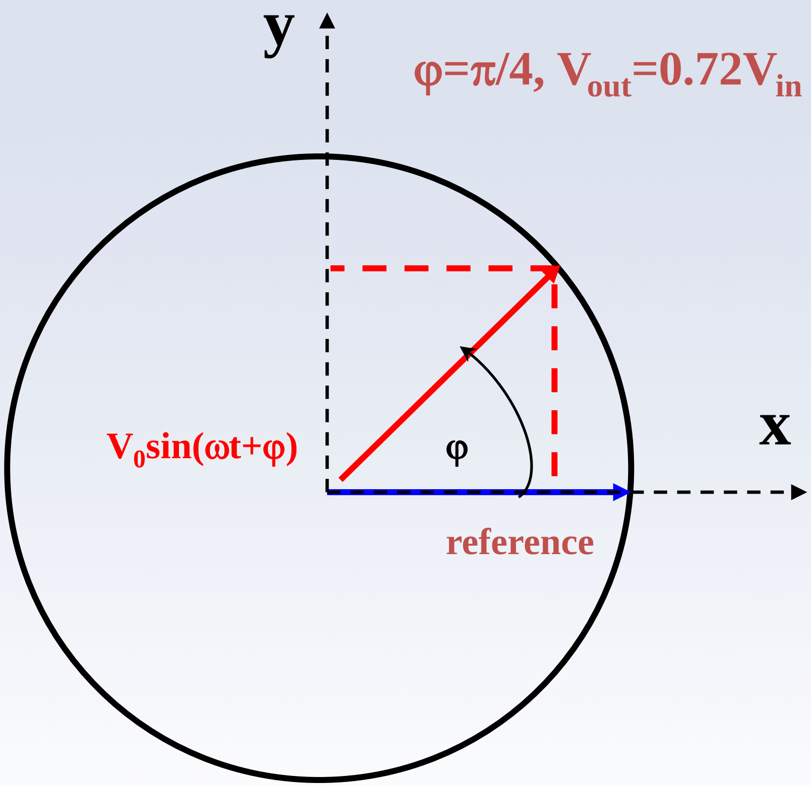


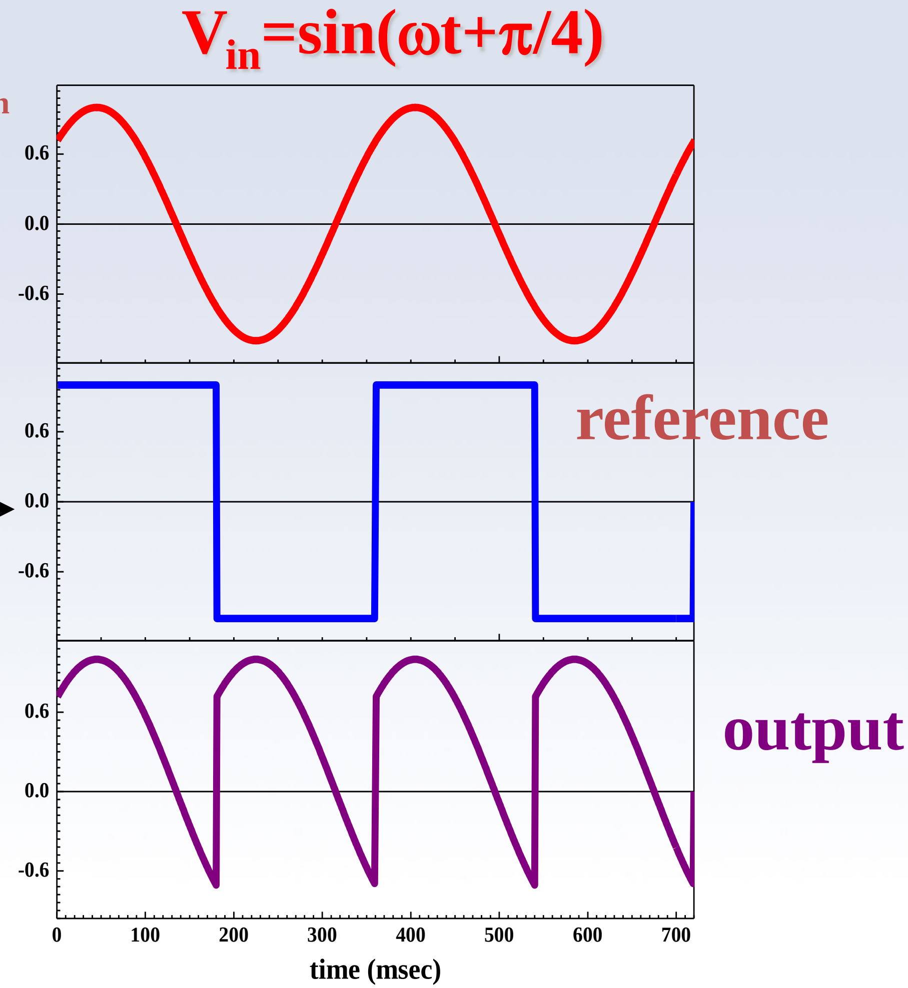
Lock-in amplifier: $f_s \neq f_r$



$V_{OUT} \rightarrow 0$

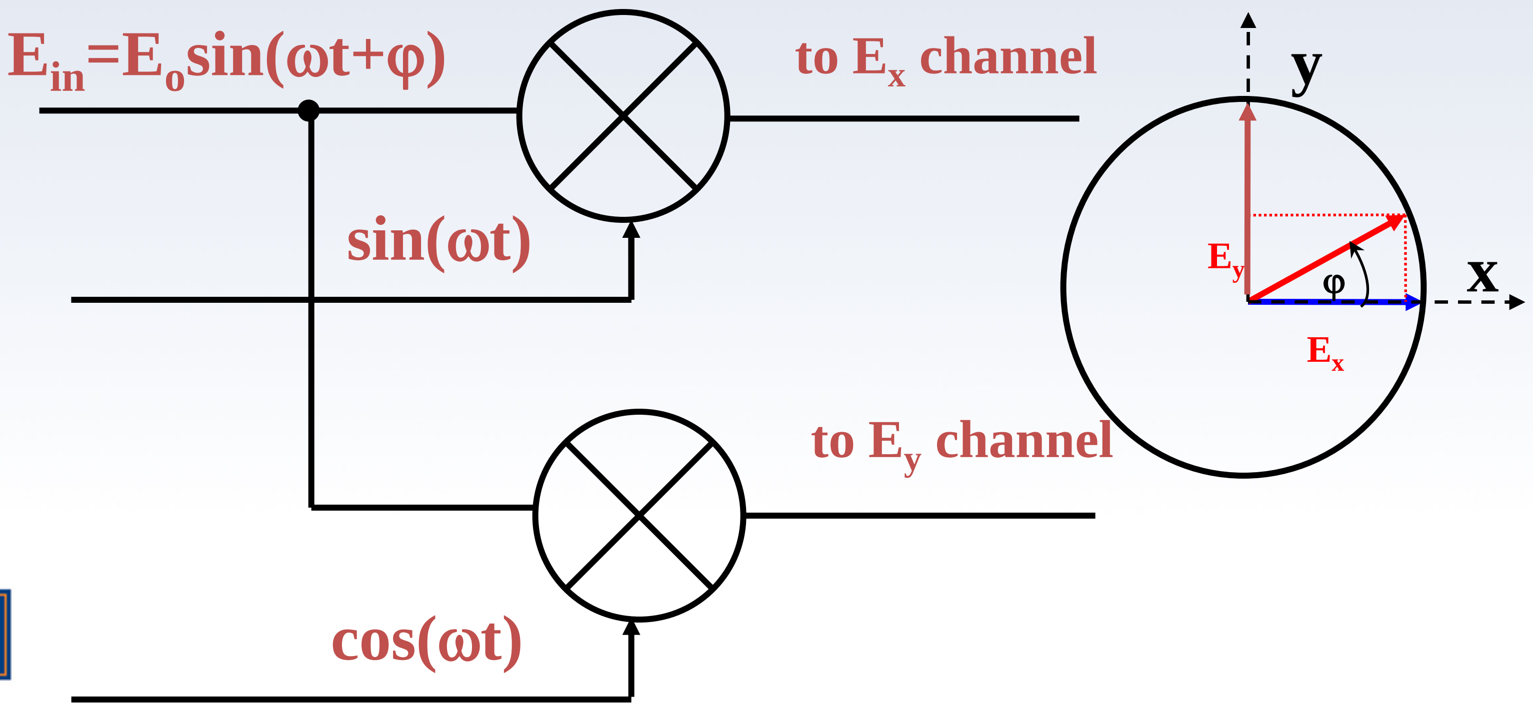
Lock-in amplifier. Phase shift.





Lock-in amplifier. Two channels demodulation.

In many scientific applications it is a great advantage to measure both components (E_x, E_y) of the input signal. We can use two lock-ins to do this or we can measure these value in two steps providing the phase shift of reference signal 0 and $\pi/2$. Much better solution is to use the lock-in amplifier equipped by two demodulators.



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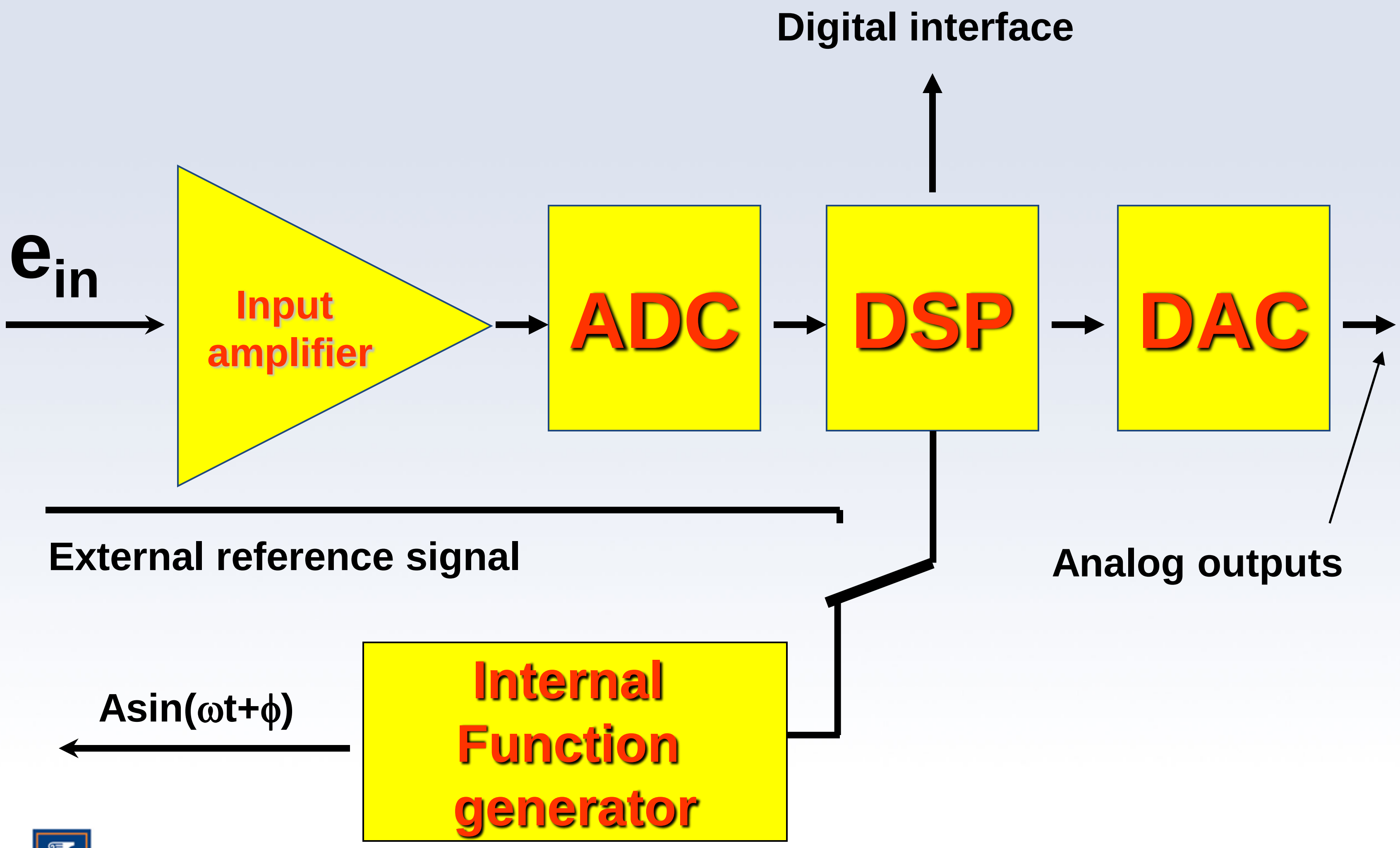
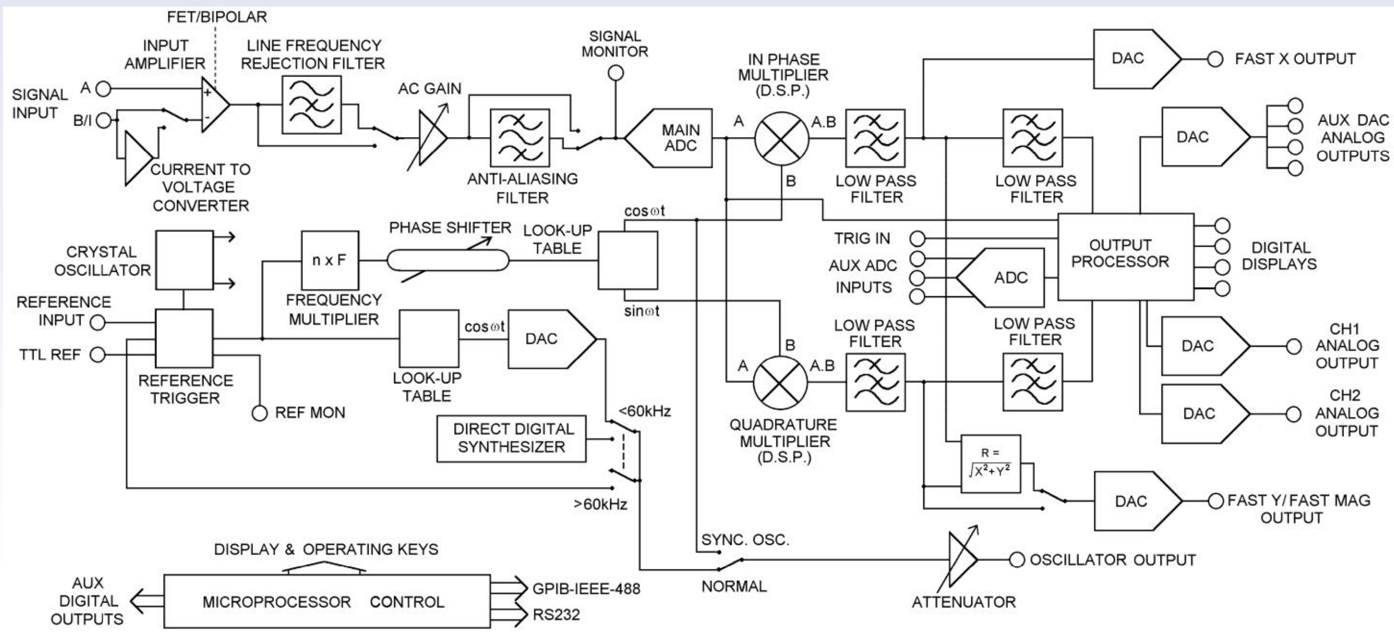
Digital Lock-in amplifier



Digital Lock-in amplifier

Digital lock-in SR830

[Lock-in demo]



Digital Lock-in amplifier. SR830.

input

time constant

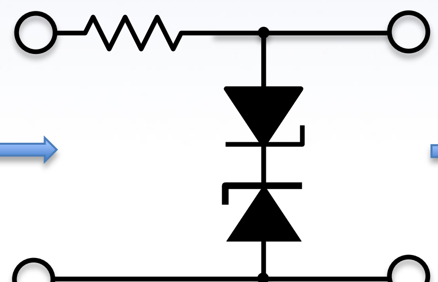
sensitivity

Channel 1

Channel 2

Sine out

Measured signal



to Lock-in

Input protecting circuit

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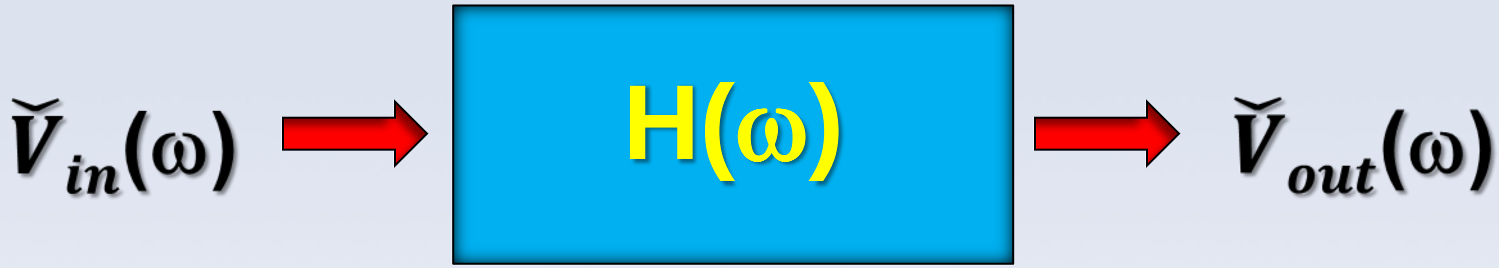
Experiments. Investigating the frequency response of circuit.

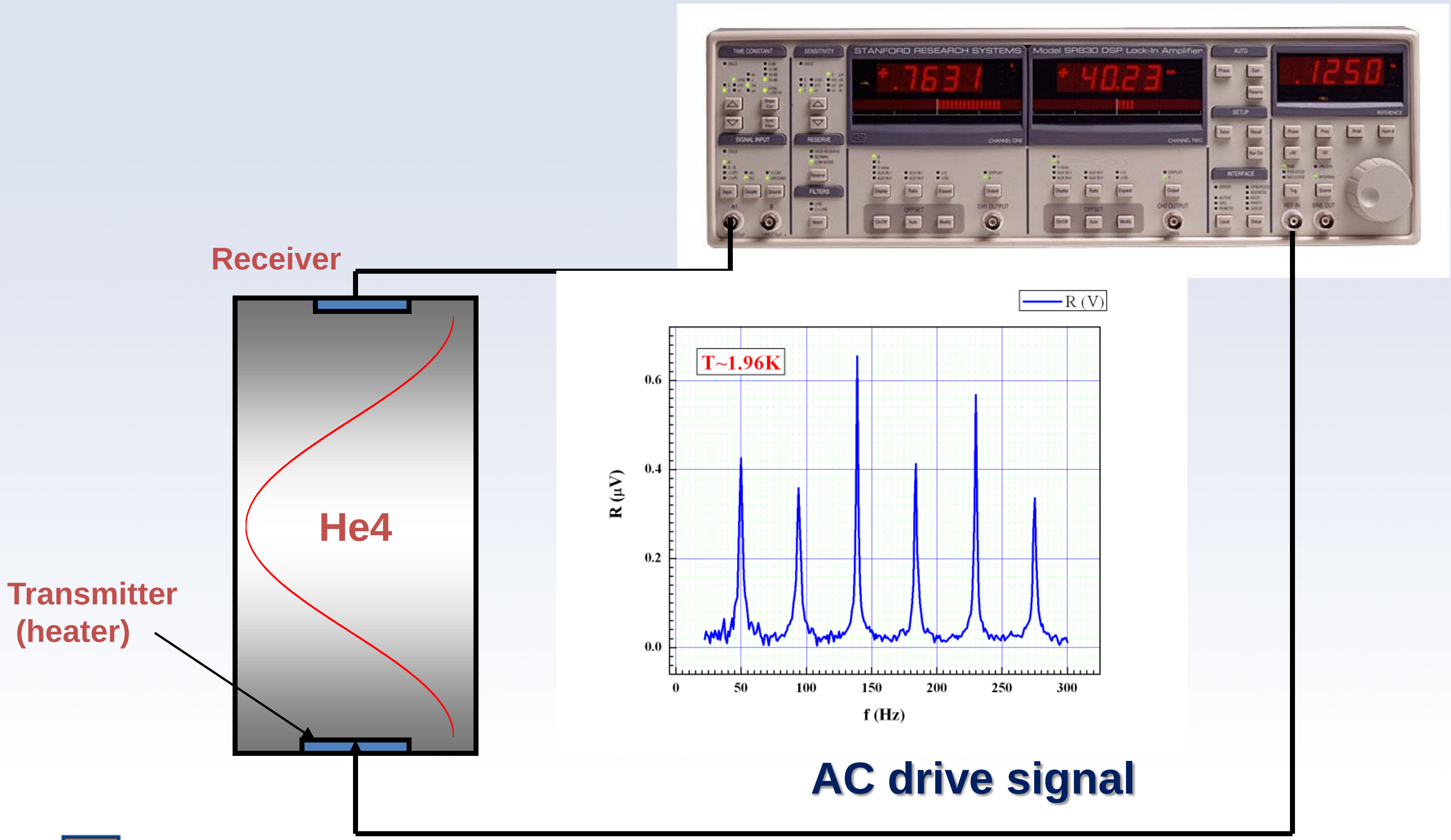


Frequency domain representation of the system

Response function → $\breve{H}(\omega) = \frac{\breve{V}_{out}}{\breve{V}_{in}}$

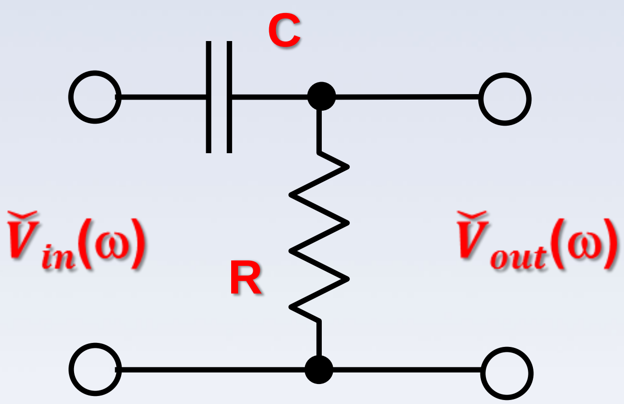
Linear engineering systems are those that can be modeled by linear differential equations.

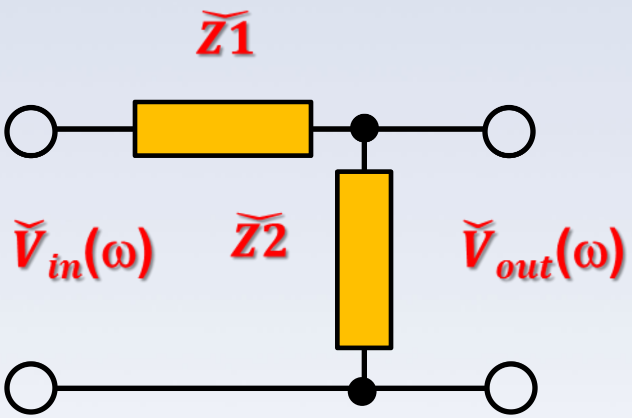
Typical application of the lock-in amplifier



Experiments. Calculation of the response function in frequency domain mode.

Example 1. High-pass filter.

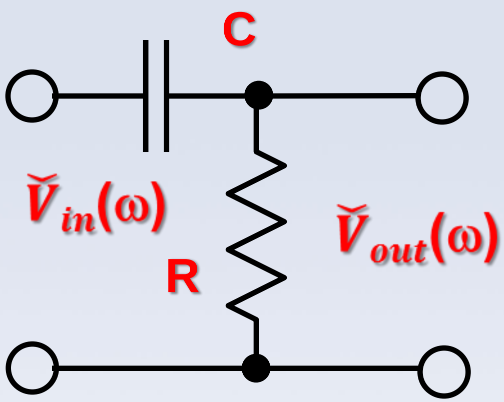




Applying the Kirchhoff Law to this simple network

$$\tilde{V}_{out}(\omega) = \tilde{H}(\omega) * \tilde{V}_{in}(\omega) = \tilde{V}_{in}(\omega)\frac{\tilde{Z}2(\omega)}{\tilde{Z}1(\omega) + \tilde{Z}2(\omega)}$$

Experiments. Calculation of the response function in frequency domain mode. High-pass filter.



Ideal case

$$\tilde{Z}_R = R$$

$$\tilde{Z}_L = j\omega L$$

$$\tilde{Z}_C = \frac{1}{j\omega C} = -\frac{j}{\omega C}$$

More realistic

$$\tilde{Z}_R = R + \ldots$$

$$\tilde{Z}_L = j\omega L + R_L$$

$$\tilde{Z}_C = \frac{1}{j\omega C} = \frac{1}{j\omega C + R_C^{-1}}$$

$$\tilde{V}_{out}(\omega) = \tilde{H}(\omega) * \tilde{V}_{in}(\omega) = \tilde{V}_{in}(\omega)\frac{\tilde{Z}2(\omega)}{\tilde{Z}1(\omega) + \tilde{Z}2(\omega)}$$

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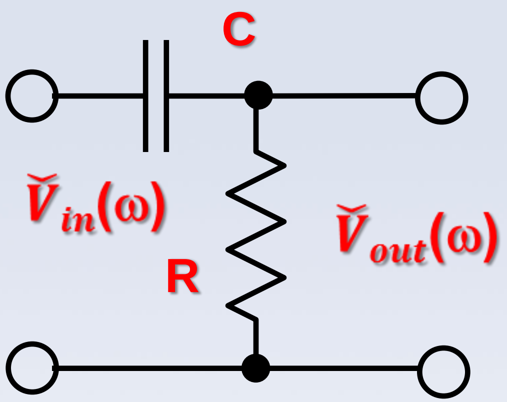
Experiments. Calculation of the response function in frequency domain mode. High-pass filter.



τ – time constant of the filter

ω_C - cutoff frequency

$$\tilde{H}(\omega) = H_R(\omega) + jH_I(\omega) = \frac{R}{R + \frac{1}{j\omega C}} = \frac{j\omega RC}{1 + j\omega RC} = \frac{j\omega\tau}{1 + j\omega\tau} = \frac{\omega\tau}{\left(1 + \omega^2\tau^2\right)}(\omega\tau + j);$$

where $\tau = RC = \omega_c^{-1}$;

$$\left|\tilde{H}(\omega)\right| = \sqrt{H_R^{\ 2} + H_I^{\ 2}} = \frac{\omega\tau}{\sqrt{1 + (\omega\tau)^2}}; \quad \theta(\omega) = \arctan\left(\frac{H_I(\omega)}{H_R(\omega)}\right) = \arctan\left(\frac{1}{\omega\tau}\right)$$

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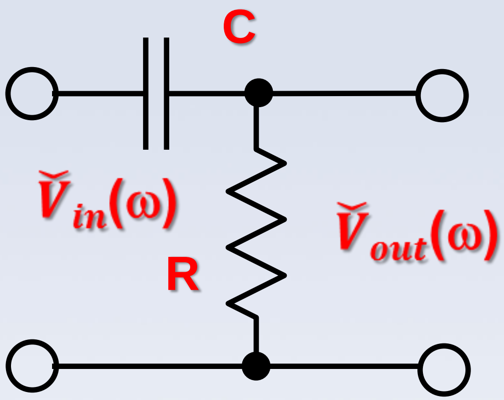
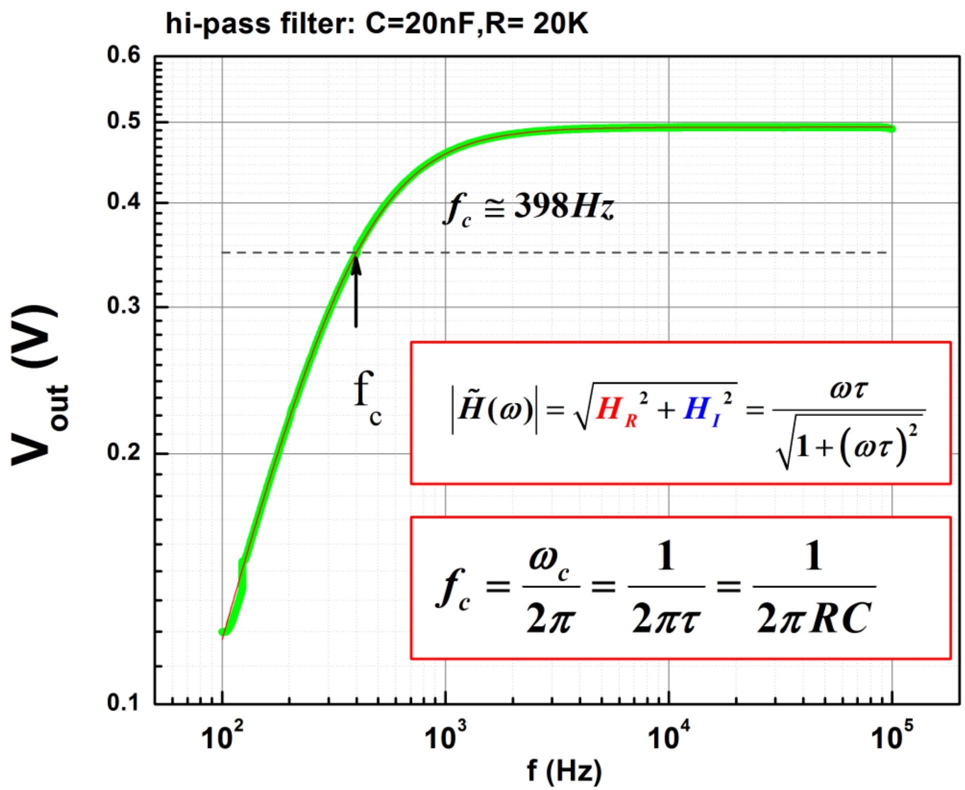
Experiments. Calculation of the response function in frequency domain mode. High-pass filter.





$$\left|\tilde{H}(\omega)\right| = \sqrt{H_R^{\,2} + H_I^{\,2}} = \frac{\omega\tau}{\sqrt{1+(\omega\tau)^2}}$$

$$f_c = \frac{\omega_c}{2\pi} = \frac{1}{2\pi\tau} = \frac{1}{2\pi RC}$$

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High-pass filter. Fitting.

100%

Tools | Format | Window | Help

- Options...
- System Variables...
- Protection
- Fitting Function Builder... F8

$$\left|\tilde{V}_{out}\right| = \left|\tilde{V}_{in}\right| \bullet \left|\tilde{H}(\omega)\right| = V_0 \bullet \frac{\omega\tau}{\sqrt{1+(\omega\tau)^2}}; \quad \tau = RC$$

Fitting parameters: V_0, τ, Voff

Parameters: V0,tau,Voff

Parameters | Constants

Param	Unit	Meaning	Fixed	Initial Value	Significant Digits
V0		?	☐	1	System
tau		?	☐	1	System
Voff		?	☐	1	System

Fitting function →

Function Body (Dependent Variables : y)

```
y=V0*2*pi*x*tau/sqrt(1+(2*pi*tau)^2)+Voff
```

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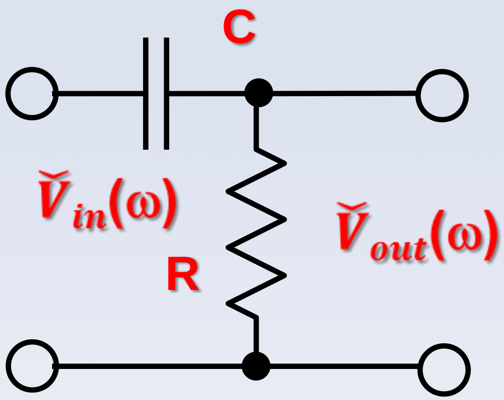
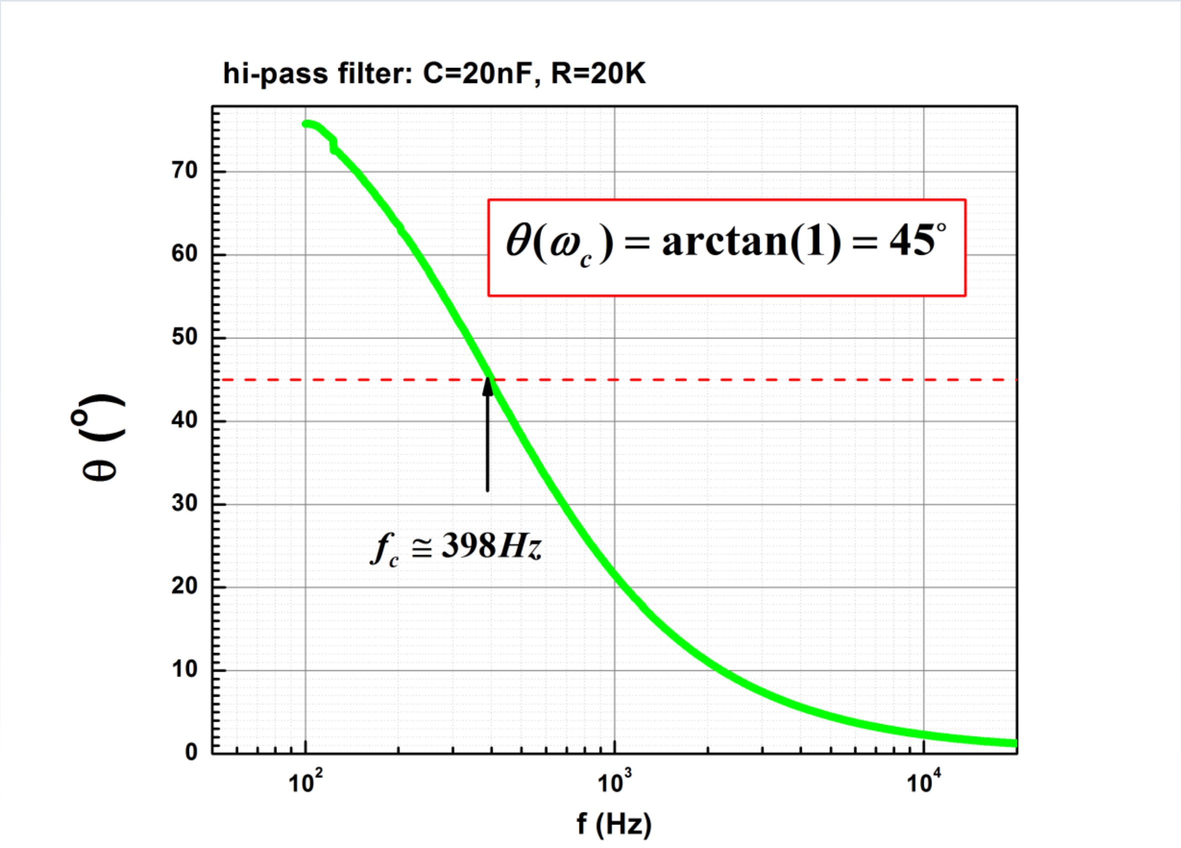
Experiments. Calculation of the response function in frequency domain mode. High-pass filter.





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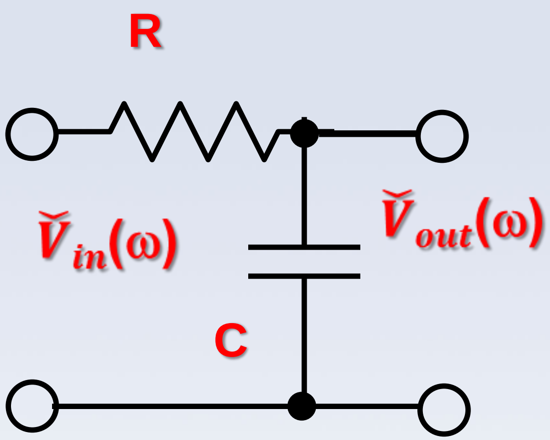
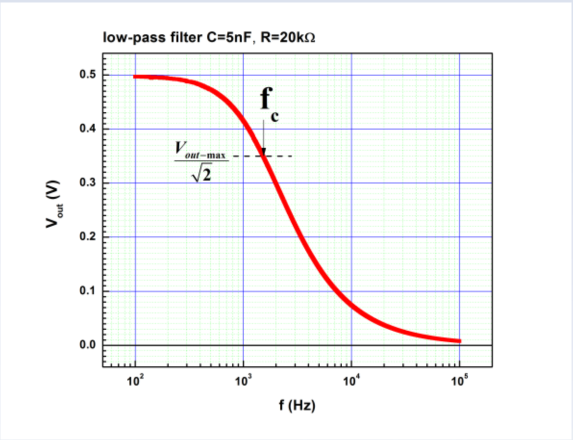
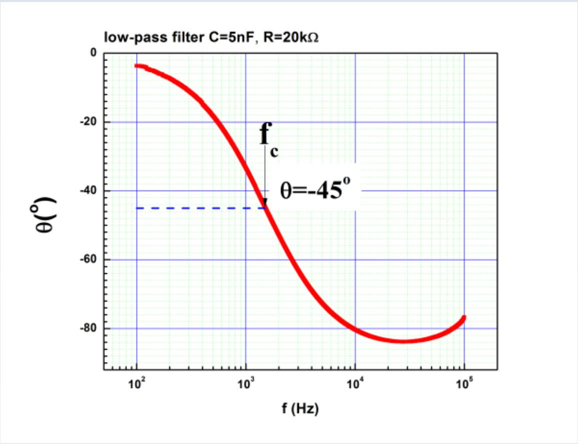
Experiments. Calculation of the response function in frequency domain mode. Low-pass filter.







$$\tilde{H}(\omega) = H_R(\omega) + jH_I(\omega) = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{1 + j\omega RC} = \frac{1}{1 + j\omega\tau} = \frac{(1 - j\omega\tau)}{(1 + \omega^2\tau^2)};$$

where $\tau = RC = \omega_c^{-1}$;

$$\left|\tilde{H}(\omega)\right| = \sqrt{H_R^{\,2} + H_I^{\,2}} = \frac{1}{\sqrt{1 + (\omega\tau)^2}}; \quad \theta(\omega) = \arctan\left(\frac{H_I(\omega)}{H_R(\omega)}\right) = -\arctan(\omega\tau)$$

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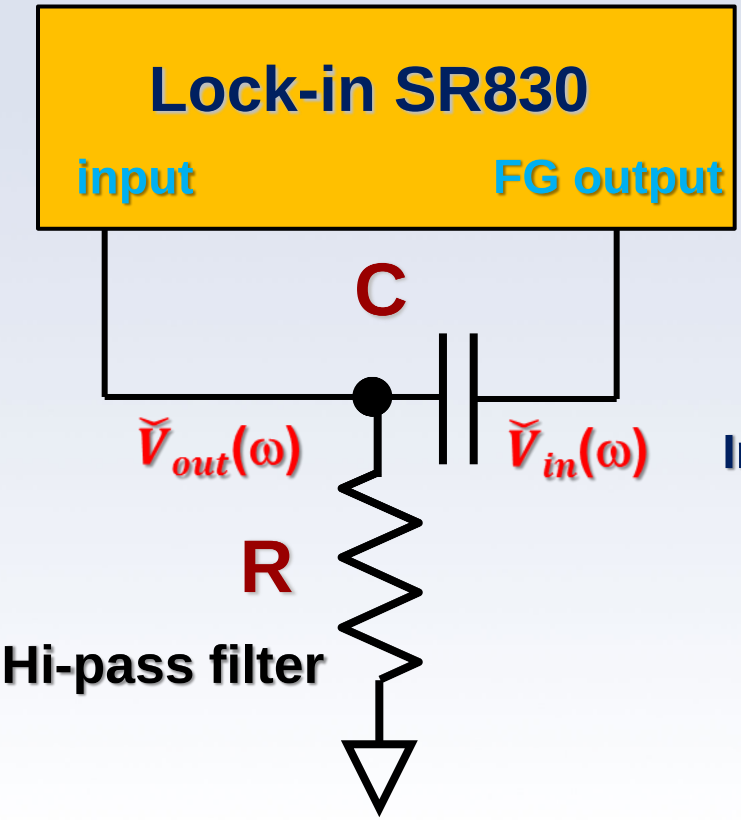
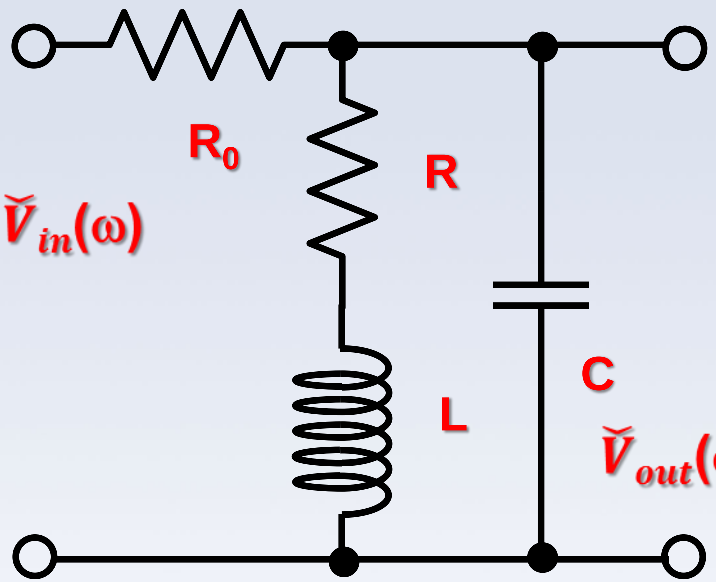
Application of the lock-in amplifier for study of the transfer function of the RLC circuit.



- Use internal reference mode
- Do measurements on harmonic no1
- Take care about time constant – should be at least ~10 times larger than period of measuring frequency
- Avoid overloading of the lock-in

Application of the lock-in amplifier for study of the transfer function of the RLC circuit.



$\check{V}_{in}(\omega)$ $\check{V}_{out}(\omega)$

if $R_0 >> R$

$$\tilde{H} \cong \left(\frac{1}{R_0}\right)\frac{R + j\omega L}{1 - \omega^2 LC + j\omega RC} = \left(\frac{1}{R_0}\right)\frac{R + j\omega L}{\left(1 - \frac{\omega^2}{\omega_0^2}\right) + j\omega RC}$$

$$\omega_o = 2\pi f = \frac{1}{\sqrt{LC}}; \quad Q = \frac{1}{\omega_0 RC} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

Application of the lock-in amplifier for study of the transfer function of the RLC circuit

$$\tilde{H} = \left(\frac{1}{R_0}\right)\frac{R + j\omega L}{\left(1 - \frac{\omega^2}{\omega_0^2}\right) + j\omega RC} = \left(\frac{R}{R_0}\right)\frac{1 + j\frac{\omega}{\omega_0}Q}{\left(1 - \frac{\omega^2}{\omega_0^2}\right) + j\frac{\omega}{\omega_0}\frac{1}{Q}}$$

$$= \left(\frac{R}{R_0}\right)\frac{1 - j\frac{\omega}{\omega_0}\left(\frac{1}{Q} - Q\left(1 - \frac{\omega^2}{\omega_0^2}\right)\right)}{\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + \left(\frac{\omega}{\omega_0}\right)^2\frac{1}{Q^2}}$$

$$\omega_o = 2\pi f = \frac{1}{\sqrt{LC}}; \qquad Q = \frac{1}{\omega_0 RC} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

Application of the lock-in amplifier for study of the transfer function of the RLC circuit

$$\tilde{H} = \left(\frac{1}{R_0}\right)\frac{R + j\omega L}{\left(1-\frac{\omega^2}{\omega_0^2}\right) + j\omega RC} = \left(\frac{R}{R_0}\right)\frac{1 + j\frac{\omega}{\omega_0}Q}{\left(1-\frac{\omega^2}{\omega_0^2}\right) + j\frac{\omega}{\omega_0}\frac{1}{Q}}$$

$$= \left(\frac{R}{R_0}\right)\frac{1 - j\frac{\omega}{\omega_0}\left(\frac{1}{Q} - Q\left(1-\frac{\omega^2}{\omega_0^2}\right)\right)}{\left(1-\frac{\omega^2}{\omega_0^2}\right)^2 + \left(\frac{\omega}{\omega_0}\right)^2\frac{1}{Q^2}}$$

fitting pars: ω_o; Q, **and** $\frac{R}{R_0}$ **(scaling coefficient)**

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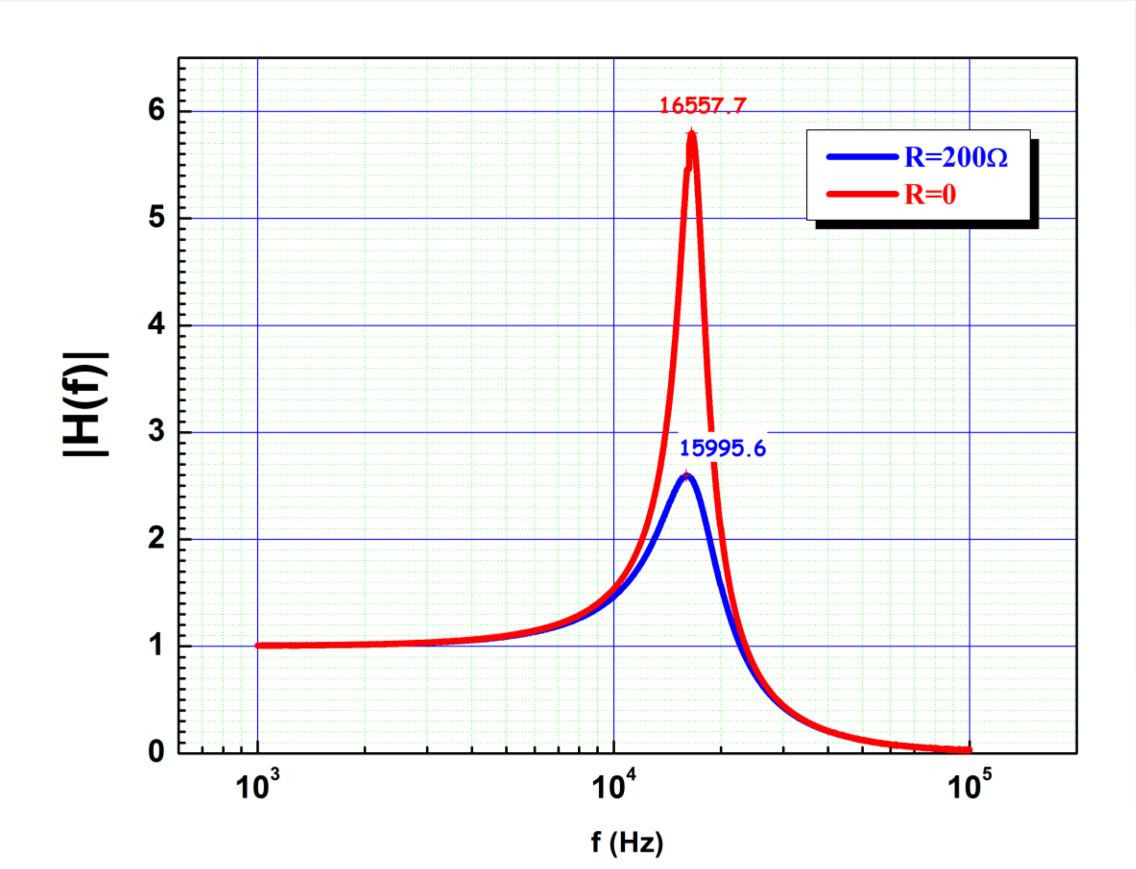
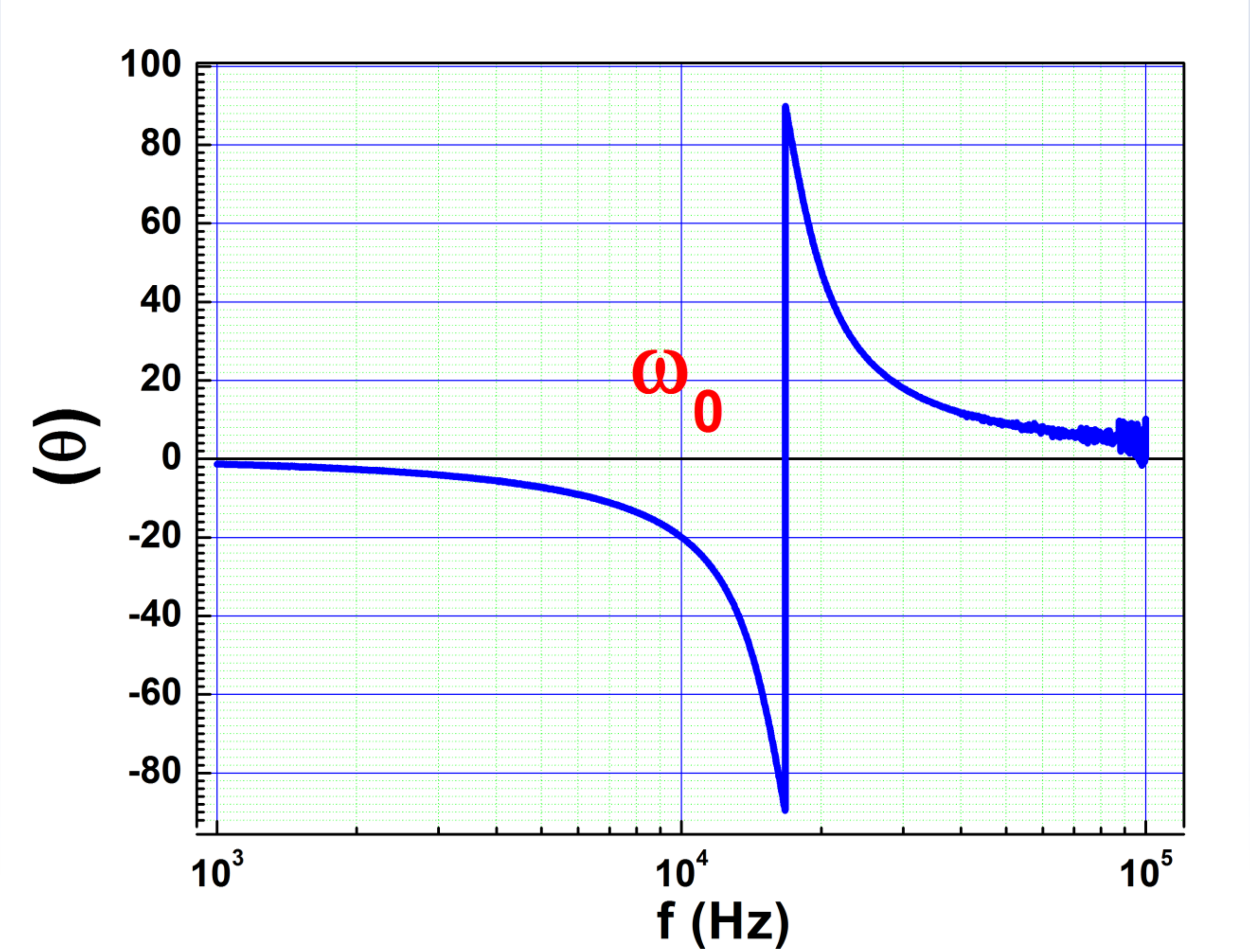
Application of the lock-in amplifier for study of the transfer function of the RLC circuit.



The resonance curves obtained on RLC circuits with two different damping resistors

Application of the lock-in amplifier for study of the transfer function of the RLC circuit.



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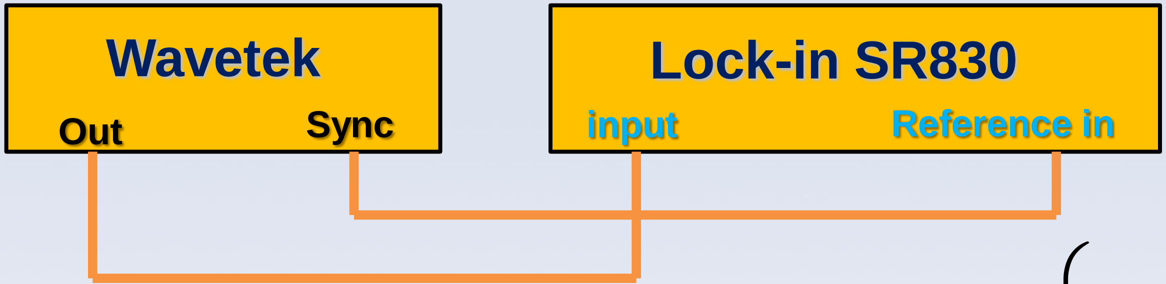
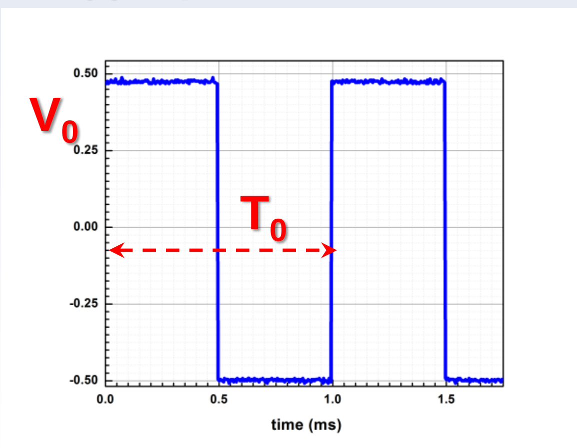
From time domain to frequency domain. Experiment.



$$V = V_0 \left(0 < t \le \frac{T_0}{2} \right);$$

$$-V_0 \left(\frac{T_0}{2} < t \le T_0 \right)$$

F(t) – periodic function $F(t)=F(t+T_0)$:



Time domain pattern

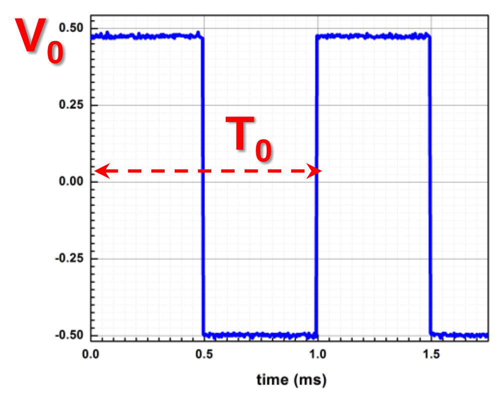
Frequency domain ?

$$a_n = \frac{2}{T_0} \int_0^{T_0} F(t) \cos\left(\frac{2\pi n t}{T_0} \right) dt;$$

$$b_n = \frac{2}{T_0} \int_0^{T_0} F(t) \sin\left(\frac{2\pi n t}{T_0} \right) dt;$$

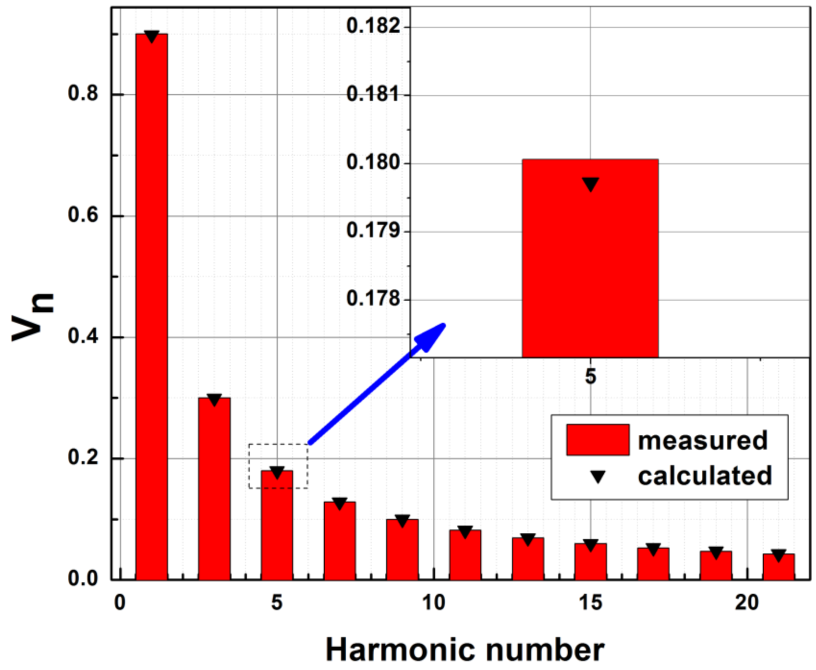
$$a_0 = \frac{2}{T_0} \int_0^{T_0} F(t) dt$$

From time domain to frequency domain. Experiment with SR830. Results.



Time domain

Spectrum measured by SR 830 lock-in amplifier



Frequency domain

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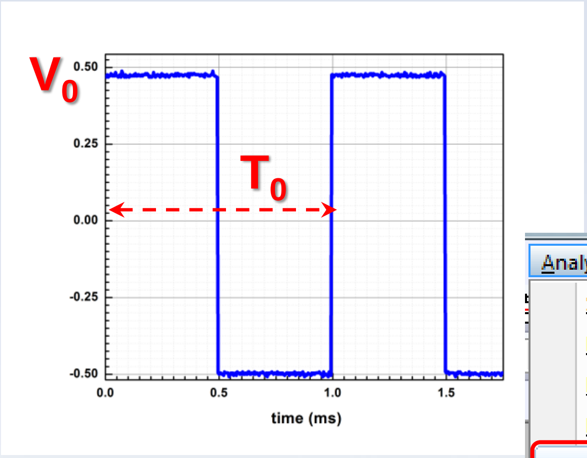
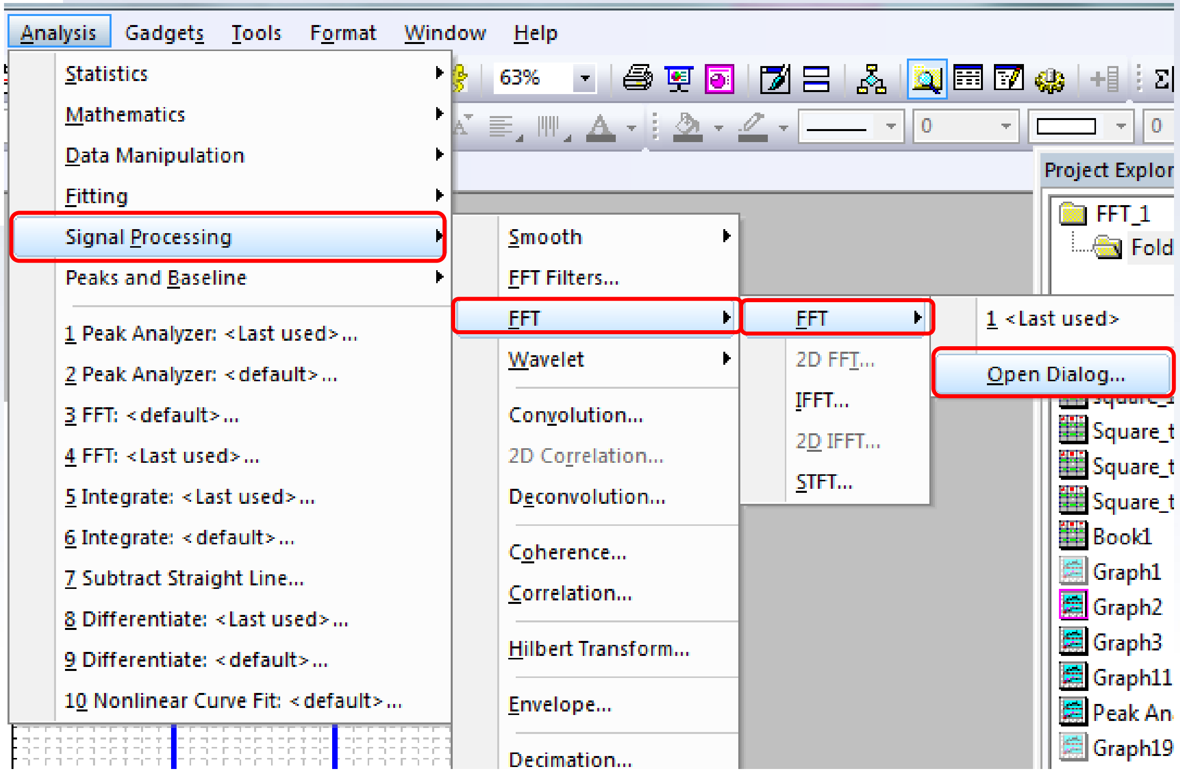
From time domain to frequency domain. FFT using Origin. Results.



Time domain taken by Tektronix scope

Data file can be used to convert time domain to frequency domain



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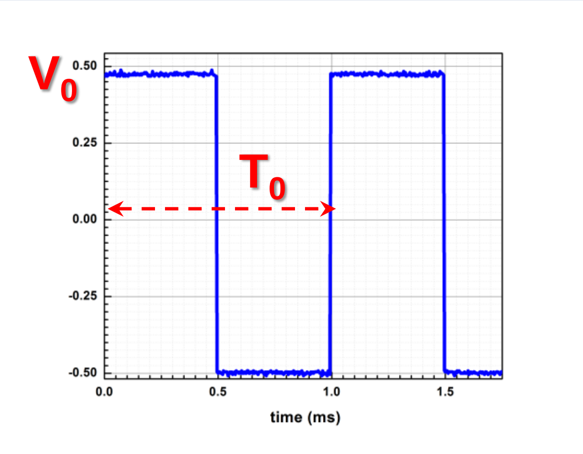
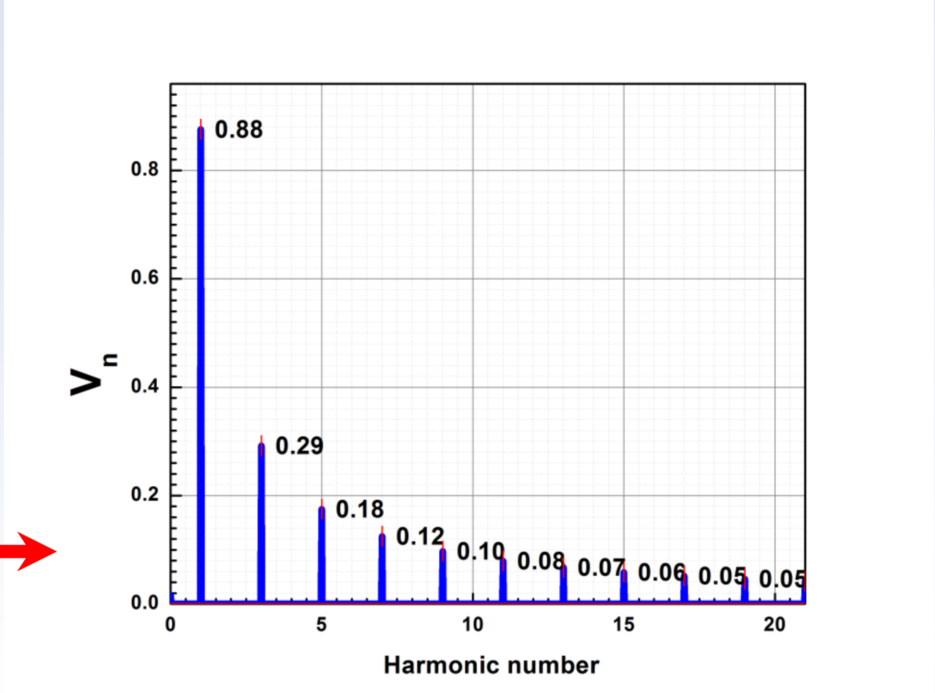
From time domain to frequency domain. FFT using Origin. Results.



Time domain taken by Tektronix scope

Data file can be used to convert time domain to frequency domain



Spectrum calculated by Origin. Accuracy is limited because of the limited resolution of the scope

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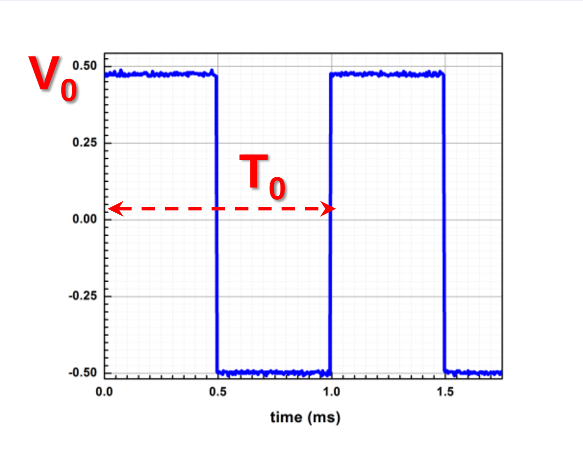
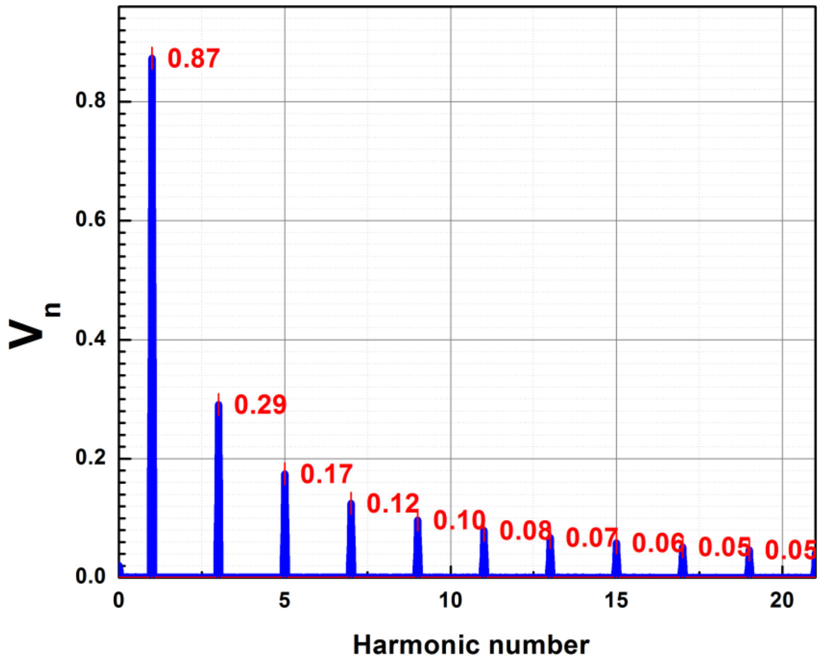
From time domain to frequency domain. Using of the Math option of the scope.



Time domain taken by Tektronix scope

Spectrum calculated by Tektronix scope.



Accuracy is limited because of the limited resolution of the scope

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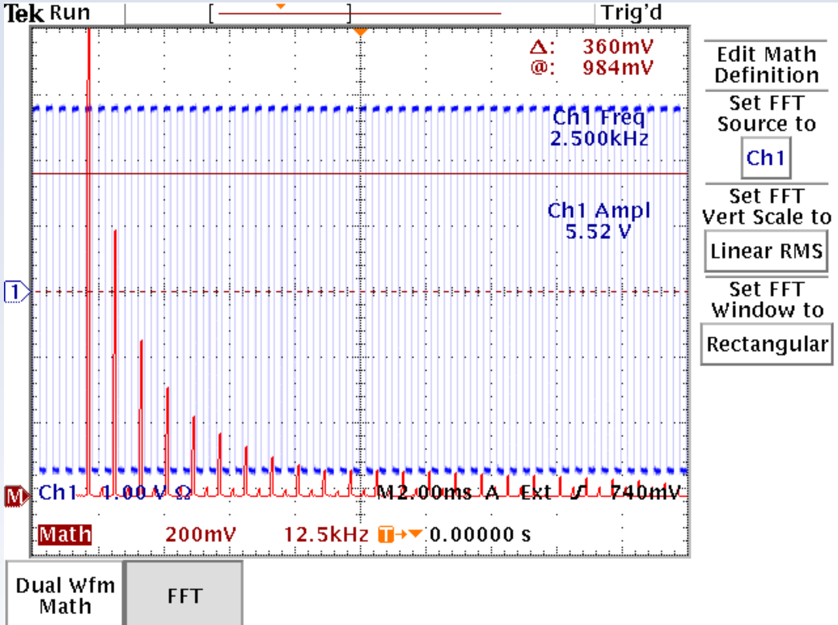
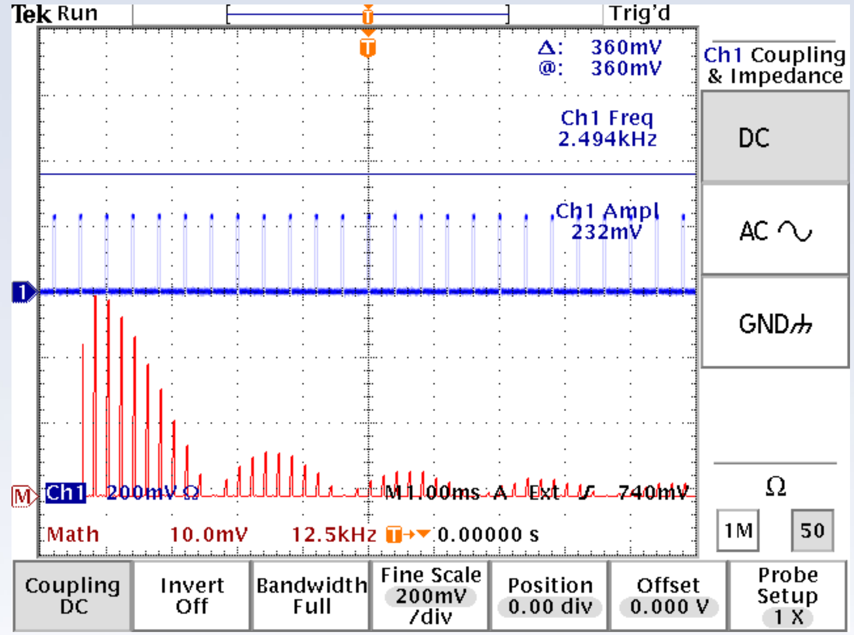
From time domain to frequency domain. Using of the Math option of the scope.



Spectrum of the square wave signal



Spectrum of the pulse signal

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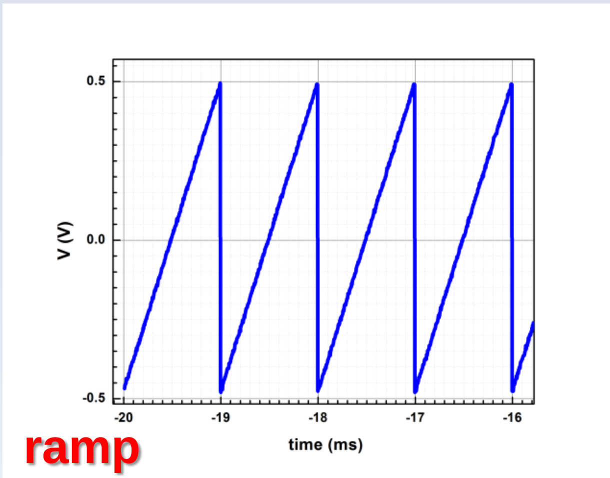
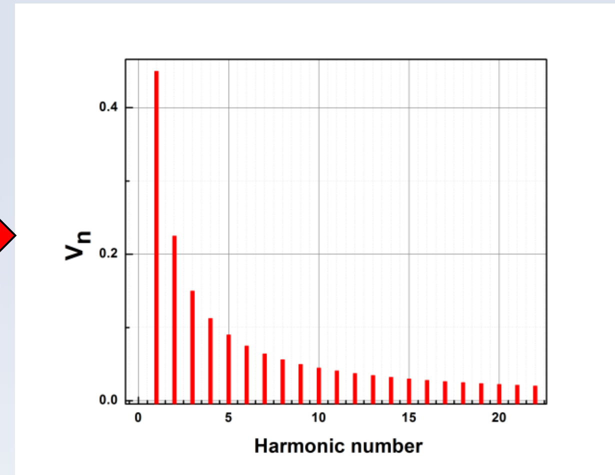
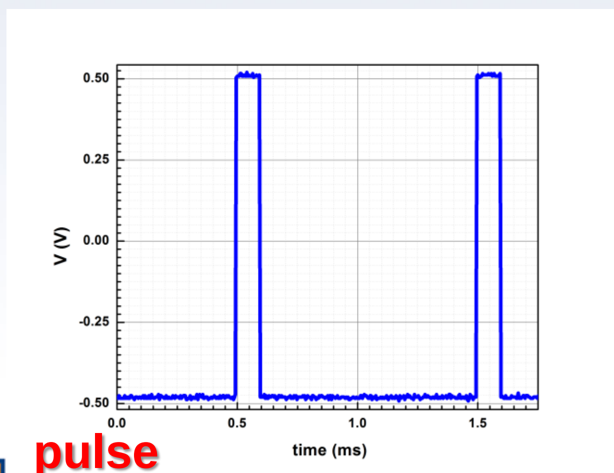
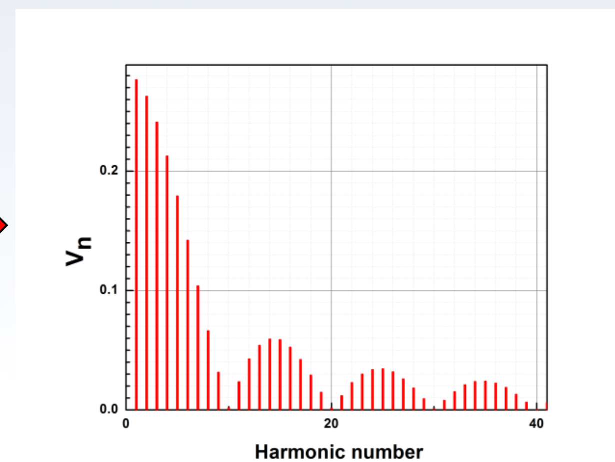
From time domain to frequency domain. Different waveforms. Lock-in data.









Appendix #1

Reminder: please submit the reports by e-mail in pdf or MsWord format. pdf is strongly preferable!

L1_lab2_student1

Lab section → L1; Lab number → lab2; Your name → student1

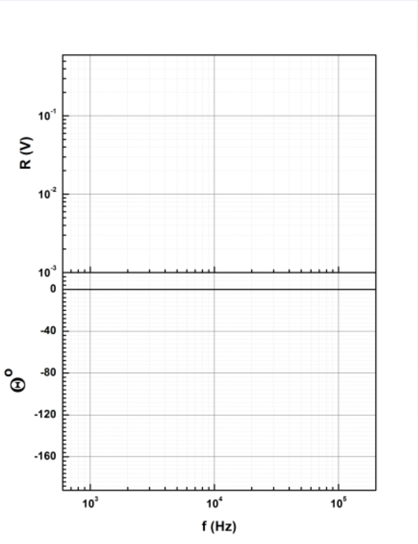
Preferable file name

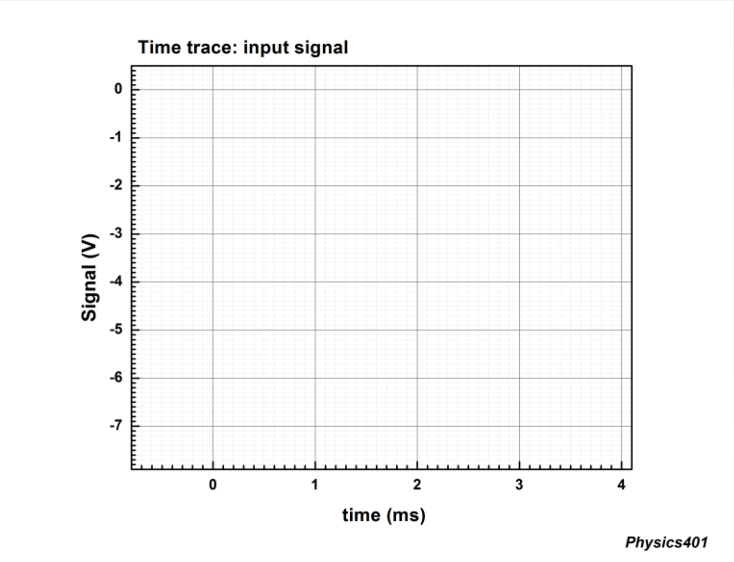
Link for uploading: https://my.physics.illinois.edu/courses/upload/

Appendix #2

Origin templates for the Lab are available in:

\\Phyaplportal\PHYCS401\Common\Origin templates\frequency domain analysis





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References:

1. John H. Scofield, "A Frequency-Domain Description of a Lock-in Amplifier" *American Journal of Physics* 62 (2) 129-133 (Feb. 1994).
2. *Steve Smith* "The Scientist and Engineer's Guide to Digital Signal Processing" *copyright ©1997-1998 by Steven W. Smith. For more information visit the book's website at: www.DSPguide.com" **

- You can find a soft copy of this book in:
- *\\Phyaplportal\PHYCS401\Experiments\DSP and FFT*

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